Scientific Computing I
(Wissenschaftliches Rechnen I)

Winter term 2019/20

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6^{th} exercise sheet

Submission on November 21, before the lecture

Exercise 1. (2+1+1+2 = 6 points)

In this exercise we consider another important reference element, the hypercube $[0,1]^d$. The C++-based finite element code `deal.II`, e.g., utilizes those type of reference elements.
On the reference interval $[0,1]$, we have defined the two functions

$$\phi_1(x) := x, \qquad \phi_2(x) := 1 - x,$$

which build a nodal basis of the polynomials of degree ≤ 1 on $[0,1]$.
For $d \in \{2,3\}$ we consider the reference cube $Q_{\text{ref}} := [0,1]^d \subset \mathbb{R}^d$. We call a function $f\colon \mathbb{R}^d \longrightarrow \mathbb{R}$ *affine multilinear*, if it is affine linear with respect to each component.

a) Let $\mathcal{M}(Q_{\text{ref}})$ be the space of affine multilinear functions defined on Q_{ref}. Show that $\mathcal{M}(Q_{\text{ref}})$ has dimension 2^d, and find a basis for $\mathcal{M}(Q_{\text{ref}})$ which is nodal with respect to the corners of Q_{ref}.

Hint: Use the structure of Q_{ref}, i.e. $Q_{\text{ref}} = [0,1] \times \ldots \times [0,1]$ and construct the basis functions with help of ϕ_1 and ϕ_2.

b) Characterize those sets $Q \subset \mathbb{R}^d$ such that there is an invertible affine linear map $B\colon \mathbb{R}^d \longrightarrow \mathbb{R}^d$ with $Q = B(Q_{\text{ref}})$.

Note: This is a crucial difference compared to tetrahedral finite elements: Every tetrahedron can be identified via an affine linear map with the reference tetrahedron.

c) Given points $x_1, \ldots, x_{2^d} \in \mathbb{R}^d$, show that there exists a map $B \in \mathcal{M}(Q_{\text{ref}})^d$, i.e. an affine multilinear map $\mathbb{R}^d \longrightarrow \mathbb{R}^d$, that maps the corners of Q_{ref} (in prescribed order) to $x_1, \ldots, x_{2^d}$. Show that such B transforms edges of Q_{ref} into subsets of straight lines and the facets of Q_{ref} into subsets of flat hyperplanes.

d) Let $\Omega \subset \mathbb{R}^d$ be a domain such that $\Omega = \bigcup_{i=1}^{N_Q} Q_i$ with invertible $B_i \in \mathcal{M}(Q_{\text{ref}})^d$ such that $Q_i = B_i(Q_{\text{ref}})$ for $i = 1, \ldots, N_Q$ and such that $Q_i \cap Q_j$ is either empty, a shared corner, a shared edge or a shared facet of both Q_i and Q_j for all $i,j = 1, \ldots, N_Q$. The corners of the Q_i are denoted by x_j, $j = 1, \ldots, N_x$.

Show that a function $\varphi\colon \Omega \longrightarrow \mathbb{R}$ such that $\varphi|_{Q_i} \circ B_i \in \mathcal{M}(Q_{\text{ref}})$ for all $i = 1, \ldots, N$ is in $H^1(\Omega)$ and determine a nodal basis for the space of all such functions.

Exercise 2. (3 + 2 + 3 = 8 points)

On the two-dimensional reference triangle $T_{\mathrm{ref}} \subset \mathbb{R}^2$,

$$T_{\mathrm{ref}} := \{(x_1, x_2) \in \mathbb{R}^2 \colon\ x_1 + x_2 \le 1, \quad x_i \ge 0 \quad \forall i = 1, 2\},$$

we consider the space of polynomials of degree $\le n$, i.e.

$$P_n(T_{\mathrm{ref}}) := \operatorname{span}\{x_1^{\alpha_1} x_2^{\alpha_2} \colon\ \alpha_1 + \alpha_2 \le n\},$$

and the point set

$$Z := \left\{ z = \left(\frac{\beta_1}{n}, \frac{\beta_2}{n}\right) \in T_{\mathrm{ref}} \colon\ \beta_1, \beta_2 \in \mathbb{N}_0 \right\}$$

a) Show that there exists a nodal basis of $P_n(T_{\mathrm{ref}})$ with respect to the point set Z. Further, show that for $B \in \mathbb{R}^{2\times 2}$ and $p \in P_n(T_{\mathrm{ref}})$ the function $p \circ B$ is a polynomial of degree $\le n$, again.

b) Show that the values of some $p \in P_n(T_{\mathrm{ref}})$ on an edge of T_{ref} are uniquely determined by its values on those points of Z belonging to this edge or facet. Describe a global nodal basis for an H^1-conforming finite element space based on this reference element.

c) Is this global space even H^2-conforming?

Programming exercise 1. (3 + 3 = 6 points)

On a domain $\Omega \subset \mathbb{R}^2$ we consider the problem

$$\begin{aligned} -\operatorname{div}(a\nabla u) + cu &= f && \text{on } \Omega, \\ u &= 0 && \text{on } \Gamma_D, \\ a\partial_n u &= g && \text{on } \Gamma_N, \\ \partial_n u &= 0 && \text{on } \partial\Omega \setminus (\Gamma_N \cup \Gamma_D), \end{aligned} \tag{1}$$

where Γ_D and Γ_N, $\Gamma_D \cap \Gamma_N = \emptyset$, denote the parts of $\partial\Omega$ on which Dirichlet and Neumann boundary conditions, respectively, are imposed. On the remaining part of the boundary natural boundary conditions hold.

Make sure that you understand how to derive the following weak formulation for (1): Find $u \in H^1_{\Gamma_D}(\Omega)$ such that

$$\int_\Omega (a\nabla u \nabla v + cuv)\, dx = \int_\Omega f v dx + \int_{\Gamma_N} g v ds \qquad \forall v \in H^1_{\Gamma_D}(\Omega).$$

Here, $H^1_{\Gamma_D}(\Omega)$ denotes the subspace of $H^1(\Omega)$ consisting of all functions with trace vanishing on $\Gamma_D \subset \partial\Omega$.

To consider such a problem we have to add an array `neumannedges` to our description of the mesh (so far given by the arrays `coordinates`, `elements`, `dirichletboundary`) that contains rowwise the edges of the mesh belonging to the Neumann boundary Γ_N encoded by the numbers of start and end node of the edge (two columns). We assume that a, c, f and g are given as functions that can be evaluated at the nodes of the triangulation.

a) Modify your finite element code from the last programming exercise such that it can be applied to the model problem (1): In the local assembly routines assume that the data a, c and f are constant on the respective element, i.e. work with

$$
\begin{aligned}
a|_T &\equiv \frac{1}{3}\left(a(a_0) + a(a_1) + a(a_2)\right), \\
c|_T &\equiv \frac{1}{3}\left(c(a_0) + c(a_1) + c(a_2)\right), \\
f|_T &\equiv \frac{1}{3}\left(f(a_0) + f(a_1) + f(a_2)\right),
\end{aligned}
$$

where a_0, a_1, a_2 denote the nodes of the element T.

For a Neumann edge E between two nodes a_0 and a_1, assume

$$
g|_E \equiv \frac{1}{2}\left(g(a_0) + g(a_1)\right)
$$

for integration.

b) Test your implementation with the following problems on $\Omega = [0,1]^2$:

- Fix the parameter $q \in \mathbb{R}$:

$$
\left.
\begin{aligned}
-\Delta u &= 2, \\
u &= 0 \quad \text{on } \Gamma_D = \{1\} \times [0,1], \\
\partial_n u &= -(q+1) \quad \text{on } \Gamma_N = \{0\} \times [0,1], \\
\partial_n u &= 0 \quad \text{on } \partial\Omega \setminus (\Gamma_D \cup \Gamma_N)
\end{aligned}
\right\} \tag{2}
$$

  with exact solution $u(x,y) = -(x-q)(x-1)$.

-

$$
\left.
\begin{aligned}
-\operatorname{div}(a\nabla u) + cu &= f, \\
u &= 0 \quad \text{on } \Gamma_D = \partial\Omega.
\end{aligned}
\right\} \tag{3}
$$

  with exact solution $u(x,y) = x(1-x)y(1-y)$ and

$$
\begin{aligned}
a(x,y) &= \exp(x+y), \\
c(x,y) &= \exp(x+y), \\
f(x,y) &= \exp(x+y)\left(y(1-y) + x(1+5y-3y^2) + x^2(y^2-3y-1)\right).
\end{aligned}
$$

Please submit the programming exercise til November 21, before the lecture, **directly to your tutor** via Email.

Exercise 3. (3 + 3 = 6 bonus points)

Now, we consider a PDE with socalled inhomogeneous Dirichlet boundary conditions:

$$\begin{aligned} -\Delta u &= f \quad \text{on } \Omega, \\ u &= h \quad \text{on } \partial\Omega. \\ \partial_n u &= 0 \quad \text{on } \Gamma_N, \end{aligned} \tag{4}$$

where $\Gamma_D \subset \partial\Omega$ is nonempty and $\Gamma_N = \partial\Omega \setminus \Gamma_D$. The inhomogeneous Dirichlet boundary conditions are given by some function $h \in L^2(\Gamma_D)$.

We assume that the boundary condition can be fulfilled by an H^2-fuction, i.e. there is some $\varphi \in H^2(\Omega)$ such that $\varphi|_{\Gamma_D} = h$ (in the sense of traces).

a) (Theoretical part) Show that (4) admits a unique solution $u \in H^1(\Omega)$ and estimate the norm of u in terms of f and φ.

Hint: Use the ansatz $u = \varphi + \hat{u}$ and derive an equation for $\hat{u}$ in $H^1_{\Gamma_D}(\Omega)$.

b) (Programming part) Modify your code of the programming exercise in such a way that it can deal with inhomogeneous Dirichlet boundary conditions as well (replace the second line of (1) by $u = h$ on Γ_D). Test with

$$-\Delta u = 0 \text{ on } [0,1]^2, \qquad u(x,y) = h(x,y) = x^5 - 10x^3y^2 + 5xy^4 \text{ on } \Gamma_D = \partial[0,1]^2.$$

Hint: The same ansatz as in a) applies on the discrete level as well.