



Institut für Numerische Simulation

Rheinische Friedrich-Wilhelms-Universität Bonn

Friedrich-Hirzebruch-Allee 7 • 53115 Bonn • Germany
phone +49 228 73-69828 • fax +49 228 73-69847
www.ins.uni-bonn.de

H.J. Flad, G. Flad-Harutyunyan

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Pseudo-Differential Algebra**

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Fundamental Solutions and Green's Functions for certain Elliptic Differential Operators from a Pseudo-Differential Algebra

Heinz-Jürgen Flad[†], Gohar Flad-Harutyunyan*,

[†]Institut für Numerische Simulation, Universität Bonn, Friedrich-Hirzebruch-Allee 7 53115 Bonn, Germany

*Fachgruppe Mathematik, RWTH Aachen University, Pontdriesch 14-16, 52062 Aachen, Germany

Correspondence: Heinz-Jürgen Flad (flad@ins.uni-bonn.de)

Gohar Flad-Harutyunyan (flad-harutyunyan@mathematik.rwth-aachen.de)

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Abstract

We have studied possible applications of a particular pseudo-differential algebra in singular analysis for the construction of fundamental solutions and Green's functions of a certain class of elliptic partial differential operators. The pseudo-differential algebra considered in the present work, comprises degenerate partial differential operators on stretched cones which can be locally described as Fuchs type differential operators in appropriate polar coordinates. We present a general approach for the explicit construction of their parametrices, which is based on the concept of an asymptotic parametrix, introduced in [8]. For some selected partial differential operators, we demonstrate the feasibility of our approach by an explicit calculation of fundamental solutions and Green's functions from the corresponding parametrices. In our approach, the Green's functions are given in separable form, which generalizes the Laplace expansion of the Green's function of the Laplace operator in three dimensions. As a concrete application in quantum scattering theory, we construct a fundamental solution of a single-particle Hamilton operator with singular Coulomb potential.

1 Introduction

In the field of partial differential equations, fundamental solution and related Green's functions are a versatile tool with a wide range of applications in mathematics, physics and engineering. Whereas the notion of a fundamental solution is uniquely defined, care has to be taken concerning the notion of a Green's function, which might differ depending on the context of its application. This is particularly true for physics where the label "Green's function" might refer, e.g., to a classical Green's function in potential theory or to a many-particle Green's function in quantum many-particle theory. Despite some differences in the various notions of a Green's function, there is an essential common feature which links them to fundamental solutions. Therefore it does not seem to be appropriate to give a single mathematically rigorous definition of a Green's function. Instead we will adopt it to the specific situation under consideration.

In the present work, we want to focus on second order linear partial differential operators of the form

$$\mathcal{A} = \sum_{|\alpha| \leq 2} a_\alpha(x) \partial^\alpha \quad (1.1)$$

in an open domain $0 \in \Omega \subseteq \mathbb{R}^n$. For such type of operators, let us briefly recall the definition of a fundamental solution and Green's function

Definition 1. A distribution $u \in \mathcal{D}'(\Omega)$ is a fundamental solution of the operator (1.1) if it satisfies the equation

$$\mathcal{A}u = \delta \quad (1.2)$$

in a distributional sense, with respect to the Dirac-distribution δ .

Fundamental solutions have been proven to exist for a large class of differential operators of the form (1.1), cf. [11, 12, 13] for a detailed account in the case of constant and smooth coefficients. In the particular case of second order elliptic differential operators with real analytic coefficients in $\Omega := B_R(0)$, e.g. the Laplace-Beltrami operator on an analytic manifold, a fundamental solution has the form

$$u(x) = \frac{1}{|x|^{n-2}}f(x) + \ln(|x|)g(x) \quad (1.3)$$

with f, g real analytic functions in Ω and $g = 0$ for n odd, cf. [9, 14, 15]. The general notion of ellipticity restricts the kernel of an elliptic operator to a finite dimensional subspace of its domain of definition. Therefore, a fundamental solution of an elliptic operator actually represents an equivalence class of distributions whose members are equal modulo elements of its kernel.

As already mentioned before, the notion of a Green's function is not well defined and we start with the most general definition appropriate for our purposes.

Definition 2. A Green's function $G \in \mathcal{D}'(\Omega)$ of the operator (1.1) is a distribution valued function $G : \Omega \rightarrow \mathcal{D}'(\Omega)$ which satisfies the distributional equation

$$\mathcal{A}G_{\tilde{x}} = \delta_{\tilde{x}} \text{ for all } \tilde{x} \in \Omega, \quad (1.4)$$

where $\delta_{\tilde{x}}$ denotes the shifted Dirac distribution, i.e., $\delta_{\tilde{x}}(f) = f(\tilde{x})$ for $f \in \mathcal{D}(\Omega)$.

In our definition of a Green's function, we abstain from specific boundary conditions and symmetry requirements which a classical Green's function is supposed to satisfy, cf. [3]. This causes a lack of uniqueness in our definition, cf. the discussion at the end of Section 3.2. The incorporation of symmetry and boundary conditions in our approach will be discussed for the specific case of the bi-Laplace operator in a forthcoming publication [2] and the general case is subject of our future work.

Before we proceed with a discussion of our specific approach to fundamental solutions and Green's functions let us briefly compare the corresponding Definitions 1 and 2, respectively. In the case of a differential operator (1.1) with constant coefficients both definitions are essentially equivalent. Let u be a regular distribution which satisfies (1.2), the corresponding Green's functions (1.4) in this case is simply given by $u(\cdot - \tilde{x})$. However, in the case of nonconstant coefficients a canonical Green's function correspond to a regular distribution $G(\cdot, \tilde{x})$, $\tilde{x} \in \Omega$, such that $G_{\tilde{x}}(f) = \int_{\Omega} G(x, \tilde{x})f(x) dx$, which cannot be derived by a simple translation from a fundamental solution. It is the intention of the present work to develop computational tools that provide fundamental solutions and Green's functions in closed form by means of power series expansions. For Green's functions, the power series, furthermore, separate the variables x and $\tilde{x}$ which is particularly suitable for numerical simulations.

In the present work, we want to discuss a novel approach for the construction of fundamental solutions and Green's functions for operators (1.1) with possibly singular coefficients a_{α} , based on

methods from singular analysis, To be precise, we allow for a single point $P \in \Omega$, w.l.o.g. located at the origin, such that $a_\alpha \in C^\infty(\Omega \setminus P)$. In order to keep control of the singular behaviour, we restrict ourselves to a specific typ of conical singularity, which can be best characterized by a transformation to an appropriate system of polar coordinates. For notational simplicity, we consider the case $\Omega = \mathbb{R}^n$. Let us introduce the cone $\mathcal{C}^\Delta(X) := \bar{\mathbb{R}}^+ \times X / (\{0\} \times X)$ with closed compact, $n-1$ -dimensional smooth base manifold X and the corresponding open stretched cone $\mathcal{C}^\wedge(X) := \mathbb{R}^+ \times X$. On $\mathcal{C}^\wedge(X)$, we choose polar coordinates (r, ϕ) , where $r \in \mathbb{R}^+$ and ϕ denotes some set of local coordinates on X . Now consider a homeomorphism $\varphi : \mathcal{C}^\Delta \rightarrow \mathbb{R}^n$ such that the tip of the cone is mapped on the origin, which induces a diffeomorphism

$$\varphi|_{\mathcal{C}^\wedge(X)} : \mathcal{C}^\wedge(X) \rightarrow \mathbb{R}^n \setminus P. \quad (1.5)$$

From this diffeomorphism, we get a representation of the operator (1.1) in polar coordinates of the form

$$\tilde{\mathcal{A}} = r^{-2} \sum_{j=0}^2 a_j(r) \left(-r \frac{\partial}{\partial r} \right)^j. \quad (1.6)$$

In the following, we will assume that $\tilde{\mathcal{A}}$ belongs to the class of degenerate partial differential operators $\text{Diff}_{\text{deg}}^2(\mathcal{C}^\wedge)$ on the open cone $\mathcal{C}^\wedge$, with coefficients $a_j(r) \in C^\infty(\bar{\mathbb{R}}^+, \text{Diff}^{2-j}(X))$, where $\text{Diff}^{2-j}(X)$, $j = 0, 1, 2$, denotes partial differential operators with smooth coefficients, at most of order $2-j$, on the base X of the cone. The preceeding assumptions concerning $\tilde{\mathcal{A}}$ are required by the pseudo-differential algebra to be discussed below. For a partial differential operator (1.6), which additionally satisfies the ellipticity conditions of the pseudo-differential algebra, cf. Appendix D, a parametrix exists.

Let us briefly outline the underlying ideas and essential features of the pseudo-differential calculus employed in the present work. For a detailed exposition, we refer to the monographs [4, 10, 20]. In general, a pseudo-differential algebra can be considered as an extension of a corresponding algebra of partial differential operators such that the extended algebra contains parametrices of its elliptic elements. Roughly speaking, a parametrix should be considered as a pseudo-inverse of an elliptic partial-differential operator. Ellipticity comprises the Fredholm property, which means that the operator has a finite dimensional kernel and cokernel. Under such a premise, the existence of a parametrix $\mathcal{P}$ can be proven, which satisfies the equations

$$\mathcal{P} \mathcal{A} = \mathcal{I} + \mathcal{K}_l \quad \text{and} \quad \mathcal{A} \mathcal{P} = \mathcal{I} + \mathcal{K}_r,$$

respectively. The parametrix can be represented as a pseudo-differential operator or as an ordinary integral operator and provides a left and right inverse modulo the compact operators $\mathcal{K}_l$ and $\mathcal{K}_r$, respectively. In the standard pseudo-differential calculus these compact operators are smoothing operators which do not encode any specific asymptotic information. In contrast to the standard pseudo-differential calculus on smooth manifolds, the singular calculus on manifolds with conical, edge or corner singularities involves compact remainders, denoted as Green operators $\mathcal{G}_l$, $(\mathcal{G}_r)$, which actually encode asymptotic information. We want to emphasize that the notions of Green's function and Green operator are different and should not be confused. As an illustrative example let us consider the action of a parametrix $\mathcal{P}$ on an equation of the type $\mathcal{A}u = f$ with possibly singular elliptic operator $\mathcal{A}$ and right hand side f . By left-multiplication with $\mathcal{P}$, the equation turns into

$$u = \mathcal{P} f - \mathcal{G}_l u.$$

Loosely speaking, the kernel function of a paramtrix $\mathcal{P}$ plays a role similar to a Green's function G , except of the presence of an remainder $\mathcal{G}_l u$ which depends on a Green operator and an unknown solution. Obviously, we don't know u , but irrespective of its particular asymptotic behaviour, the Green operator $\mathcal{G}_l$ maps it into a space with well defined asymptotic behaviour which only depends on $\tilde{\mathcal{A}}$, cf. [20] for a detailed exposition. In the present work, we actually show that by an appropriate choice of a specific weight parameter of the parameterix, its kernel function satisfies our Definition 2 of a Green's function. This follows from a property of the corresponding Green operator which maps u into the kernel of $\tilde{\mathcal{A}}$ such that $\mathcal{G}_l u$ can be extended from $C^\wedge \equiv \mathbb{R} \setminus \{0\}$ to $\mathbb{R}^n$ and its extension $(\mathcal{G}_l u)_{\text{ext}}$ satisfies the homogeneous equation $\mathcal{A}(\mathcal{G}_l u)_{\text{ext}} = 0$.

In Section 2, we present a general approach for the explicit construction of these parametrices, based on the concept of an asymptotic parametrix, introduced in Refs. [7, 8]. Within our approach, we want to extract fundamental solutions and Green's functions from the integral kernels of these parametrices. In order to demonstrate its feasibility, we provide in Sections 3 and 4 some explicit calculations of fundamental solutions and Green's functions from integral kernels of parametrices for some selected partial differential operators.

An important point which deserves our attention is the fact, that fundamental solutions and Green's functions are by definition distributions in $\Omega \subseteq \mathbb{R}^n$, whereas the kernel of a parametrix and derived quantities are functions on the stretched cone $\mathcal{C}^\wedge(X)$ where the origin is excluded by definition. Therefore we will make use of Hadamard's notion of a pseudofunction, cf. [21]. By the diffeomorphism $x = \varphi(r, \phi)$, cf. (1.5), a function $u(r, \phi)$ on $\mathcal{C}^\wedge(X)$ corresponds to a function $\tilde{u}(x)$ on $\mathbb{R}^n \setminus P$. A function u on $\mathcal{C}^\wedge(X)$ can be regarded as a regular distribution on Ω if $\tilde{u}$ can be identified with an element in $L^1_{\text{loc}}(\mathbb{R}^n)$, the set of equivalence classes of locally integrable functions in $\mathbb{R}^n$, and therefore with a regular distribution in $\mathcal{D}'(\mathbb{R}^n)$. We call this regular distribution the pseudofunction corresponding to u and denote it by $\text{Pf. } u$.

Let us briefly illustrate these concepts for the most prominent case in applications which is the Laplace operator Δ_3 in $\mathbb{R}^3$.

Example 1. A fundamental solution of the Laplace operator Δ_3 in $\mathbb{R}^3$ is given by $u(x) = -\frac{1}{4\pi|x|}$ and the corresponding Green's function by $G(x, \tilde{x}) = u(x - \tilde{x}) = -\frac{1}{4\pi|x - \tilde{x}|}$. Given any $f \in \mathcal{D}(\mathbb{R}^3)$ the convolution

$$g(x) := \int_{\mathbb{R}^3} G(x, \tilde{x}) f(\tilde{x}) d\tilde{x} \quad (1.7)$$

satisfies the equation $\mathcal{A}g = f$.

In spherical polar coordinates, the Laplace operator is given by

$$\tilde{\Delta}_3 := \frac{1}{r^2} \left[\left(-r \frac{\partial}{\partial r} \right)^2 - \left(-r \frac{\partial}{\partial r} \right) + \Delta_{S^2} \right]$$

and its fundamental solution can be expressed as

$$u(x) = \text{Pf. } -\frac{1}{4\pi r}.$$

The Green's function $G(x, \tilde{x})$ of the Laplace operator, also known as Newton or Coulomb potential in the physics literature, can be represented in a separable form by the Laplace expansion, i.e.,

$$G(\cdot, \tilde{x}) := \begin{cases} \text{Pf. } G(\cdot | \tilde{r}, \tilde{\phi}) & \text{for } \tilde{x} = \varphi(\tilde{r}, \tilde{\phi}) \neq 0 \\ \text{Pf. } \lim_{\tilde{r} \rightarrow 0} G(\cdot | \tilde{r}, \tilde{\phi}) & \text{for } \tilde{x} = 0 \end{cases}$$

with

$$G(r, \phi | \tilde{r}, \tilde{\phi}) = - \sum_{\ell=0}^{\infty} \frac{r_{<}^{\ell}}{r_{>}^{\ell+1}} \frac{1}{2\ell+1} \sum_{m=-\ell}^{\ell} (-1)^m Y_{\ell,m}(\theta, \varphi) Y_{\ell,-m}(\tilde{\theta}, \tilde{\varphi}) \quad (1.8)$$

and $r_{<} := \min\{|x|, |\tilde{x}|\}$ and $r_{>} := \max\{|x|, |\tilde{x}|\}$, respectively, where (1.8) corresponds to a function on $\mathcal{C}^\wedge(S^2) \times \mathcal{C}^\wedge(S^2)$.

In order to achieve a separation of variables, the Laplace expansion is done with respect to the eigenfunctions of the Laplace-Beltrami operator on the sphere S^2 , so called spherical harmonics, $Y_{\ell,m}$, $\ell = 0, 1, \dots$ and $m = -\ell, \dots, \ell$. By taking the limit $\tilde{r} \rightarrow 0$ in (1.8) one can easily recover the fundamental solution u . The Laplace expansion greatly facilitates the computation of the convolution (1.7) and is therefore of great practical significance in computational physics and chemistry.

Our change of perspective from cartesian to polar coordinates is a key component of the present work and is motivated by the fundamental solutions (1.3) for differential operators of the form (1.1) with real analytic coefficients and the Laplace expansion of the Green's function (1.8), where $\mathbb{R}^3$ has been replaced by the cone $\mathcal{C}^\Delta := \mathbb{R}^+ \times S^2 / (\{0\} \times S^2)$ with base S^2 on which the Laplace-Beltrami operator has a pure point spectrum and the eigenfunctions form a complete basis in $L_2(S^2)$.

1.1 Outline of our approach for the Laplace operator

Before we enter into a general discussion for second order differential operators in the cone algebra, we want to exemplify our approach for the Laplace operator in $\mathbb{R}^n$, with $n \geq 3$ ¹. Following our discussion in Section 1, there are two different types of representation for the Laplace operator in n dimensions, depending on whether one considers it in $\mathbb{R}^n$ with respect to Cartesian coordinates

$$\Delta_n := \sum_{j=1}^n \partial_j^2 \quad (1.9)$$

or some kind of spherical polar coordinates defined on the stretched cone $\mathcal{C}^\wedge := \mathbb{R}^+ \times S^{n-1}$ with base S^{n-1} . In spherical polar ($n = 3$), and hyperspherical polar ($n \geq 3$) coordinates, the Laplace operator is represented by

$$\tilde{\Delta}_n := \frac{1}{r^2} \left[\left(-r \frac{\partial}{\partial r} \right)^2 - (n-2) \left(-r \frac{\partial}{\partial r} \right) + \Delta_{S^{n-1}} \right]. \quad (1.10)$$

where $\Delta_{S^{n-1}}$ denotes the Laplace-Beltrami operator on the $n-1$ sphere S^{n-1} . The Laplace-Beltrami operator $\Delta_{S^{n-1}}$ has a pure point spectrum with eigenvalues $\lambda_\ell = -\ell(\ell + n - 2)$, $\ell = 0, 1, 2, \dots$. For $n = 3$ each eigenvalue λ_ℓ has multiplicity $2\ell + 1$ and for $n > 3$ its multiplicity is given by

$$\frac{(\ell + n/2 - 1) \prod_{j=1}^{n-3} (\ell + j)}{(n/2 - 1) \cdot (n - 3)!},$$

cf. [19] for further details. In the following, we denote by P_ℓ , $\ell = 0, 1, 2, \dots$, the projection operators from $L^2(S^{n-1})$ on the corresponding eigenspaces of the eigenvalues $-\ell(\ell + n - 2)$. Herewith, we can form the spectral resolution

$$\Delta_{S^{n-1}} = - \sum_{\ell=0}^{\infty} \ell(\ell + n - 2) P_\ell \quad (1.11)$$

¹The case $n = 2$ is different, according to our discussion in Section 2.1

which is crucial for the following considerations.

The basic idea of our approach is to consider (1.10) as an element of an operator algebra which enables the construction of a parametrix. This can be achieved within the pseudo-differential cone algebra, developed by Schulze and collaborators, cf. the monographs [4, 10, 20]. Existence of a parametrix for a partial differential operator is intimately connected with its ellipticity. In the framework of the pseudo-differential calculus, considered in the present work, the notion of ellipticity involves a whole hierarchy of symbols associated to a partial differential operator, cf. Chapter 10 of [10] for a detailed discussion. For unbounded domains, e.g. $\mathcal{C}^\wedge$, one has to take into account the exit behaviour to infinity, as a result, the Laplace operator $\tilde{\Delta}_n$ is not elliptic in $\mathcal{C}^\wedge$, because it fails to satisfy the elliptic exit condition. It is only the shifted Laplacian $\tilde{\Delta}_n - \kappa^2$, discussed in Section 3, which satisfies all ellipticity conditions. For the formal construction of a parametrix outlined below, the exit condition is not essential and does not affect the final result. A rigorous justification for disregarding it will be given in Section 4, where we consider a fundamental solution and Green's function for shifted Laplacian and recover the results given below by taking the limit $\kappa \rightarrow 0$.

In order to construct a parametrix, we have to represent (1.10) as Mellin type pseudo-differential operator, i.e.,

$$\tilde{\Delta}_n u = r^{-2} \text{op}_M^{\gamma - \frac{n-1}{2}}(h)u$$

for $u \in \tilde{\mathcal{D}}(\mathcal{C}^\wedge)$, here $\tilde{\mathcal{D}}(\mathcal{C}^\wedge) := \{\varphi^*g \mid g \in \mathcal{D}(\mathbb{R}^n)\}$, where φ^*g denotes the pullback under the diffeomorphism (1.5), with operator valued Mellin symbol

$$\begin{aligned} h(w) &= w^2 - (n-2)w + \Delta_{S^{n-1}} \\ &= w^2 - (n-2)w - \sum_{\ell=0}^{\infty} \ell(\ell+n-2)P_\ell \end{aligned} \tag{1.12}$$

we refer, e.g., to [4, Chapter 8] for further details. For the parametrix we take the ansatz

$$\mathcal{P}u = r^2 \text{op}_M^{\gamma - \frac{n+3}{2}}(h^{(-1)}(w))u$$

and consider the operator product

$$\begin{aligned} \mathcal{P}_n \tilde{\Delta}_n &= r^2 \text{op}_M^{\gamma - \frac{n+3}{2}}(h^{(-1)}(w))r^{-2} \text{op}_M^{\gamma - \frac{n-1}{2}}(h(w)) \\ &= \text{op}_M^{\gamma - \frac{n-1}{2}}(h^{(-1)}(w+2)) \text{op}_M^{\gamma - \frac{n-1}{2}}(h(w)) \\ &= \text{op}_M^{\gamma - \frac{n-1}{2}}(h^{(-1)}(w+2)h(w)) \end{aligned}$$

The operator valued symbol of the parametrix has to satisfy the equation

$$h^{(-1)}(w+2)h(w) = 1$$

which can be solved for

$$\begin{aligned} h^{(-1)}(w) &= \frac{1}{h(w-2)} \\ &= \frac{1}{(w-2)^2 - (n-2)(w-2) + \Delta_{S^{n-1}}} \\ &= \sum_{\ell=0}^{\infty} \underbrace{\frac{P_\ell}{(w-2+\ell)(w-\ell-n)}}_{=: h_\ell^{(-1)}(w)}. \end{aligned} \tag{1.13}$$

It can be easily that the terms in the sum (1.13) have only simple poles for $n \geq 3$ but has a pole of order 2 for $n = 2$ at $\ell = 0$. It turns out, that for this particular reason, the case $n = 2$ is special, therefore we only consider the case $n \geq 3$.

Let $u \in \tilde{\mathcal{D}}(\mathcal{C}^\wedge)$, the action of the parametrix $\mathcal{P}_n$ is given by the double integral

$$r^2 \operatorname{op}_M^{\gamma - \frac{n+3}{2}}(h^{(-1)})u = r^2 \int_{\mathbb{R}} \int_0^\infty \left(\frac{r}{\tilde{r}}\right)^{-(\frac{n+4}{2} - \gamma + i\rho)} h^{(-1)}(\frac{n+4}{2} - \gamma + i\rho) u(\tilde{r}, \phi) \frac{d\tilde{r}}{\tilde{r}} d\rho, \quad (1.14)$$

with $d\rho := \frac{d\rho}{2\pi}$. Here we choose $2 - \frac{n}{2} < \gamma < \frac{n}{2}$ which is related to the ellipticity condition of the conormal symbol in the cone algebra. Splitting the radial integral into two parts, i.e., $\tilde{r} < r$ and $\tilde{r} > r$, one can apply Cauchy's residue theorem to the spectral resolution (1.11) of the operator valued symbol. Taking into account, that all poles in (1.11) are of order 1, one can calculate the integral along the complex line by closing the path on the right and left hand side, respectively. After some algebraic manipulations one obtains

$$\begin{aligned} \mathcal{P}_n u(r, \phi) &= - \sum_{\ell=0}^{\infty} \int_{S^{n-1}} \int_0^r \frac{\tilde{r}^\ell}{r^{\ell+n-2}} \frac{p_\ell(\phi|\tilde{\phi})}{2\ell+n-2} u(\tilde{r}, \tilde{\phi}) \tilde{r}^{n-1} d\tilde{r} \mu_{n-1}(\tilde{\phi}) d\tilde{\phi} \\ &\quad - \sum_{\ell=0}^{\infty} \int_{S^{n-1}} \int_r^\infty \frac{r^\ell}{\tilde{r}^{\ell+n-2}} \frac{p_\ell(\phi|\tilde{\phi})}{2\ell+n-2} u(\tilde{r}, \tilde{\phi}) \tilde{r}^{n-1} d\tilde{r} \mu_{n-1}(\tilde{\phi}) d\tilde{\phi}. \end{aligned}$$

Therefore, the parametrix can be represented by an integral operator

$$\mathcal{P}_n u(r, \phi) = \int_{S^{n-1}} \int_0^\infty K_n(r, \phi|\tilde{r}, \tilde{\phi}) u(\tilde{r}, \tilde{\phi}) \tilde{r}^{n-1} d\tilde{r} \mu_{n-1}(\tilde{\phi}) d\tilde{\phi}$$

with kernel function

$$K_n(r, \phi|\tilde{r}, \tilde{\phi}) = - \sum_{\ell=0}^{\infty} \frac{r_{<}^\ell}{r_{>}^{\ell+n-2}} \frac{p_\ell(\phi|\tilde{\phi})}{2\ell+n-2},$$

where $p_\ell(\phi|\tilde{\phi})$ denotes the integral kernel of the projection operator P_ℓ , $\ell = 0, 1, 2, \dots$ and $r_{<} := \min\{r, \tilde{r}\}$, $r_{>} := \max\{r, \tilde{r}\}$, respectively. For $n = 3$, using spherical coordinates $\phi = (\theta, \varphi)$, we have

$$p_\ell(\theta, \varphi|\tilde{\theta}, \tilde{\varphi}) = \sum_{m=-\ell}^{\ell} (-1)^m Y_{\ell,m}(\theta, \varphi) Y_{\ell,-m}(\tilde{\theta}, \tilde{\varphi})$$

and recover the well known Laplace expansion of the Green's function of Δ_3 , given by (1.8).

The Laplace operator is a particularly simple case because the symbol of the parametrix can be derived by direct inversion (1.13). This was only possible, because all terms which contain derivatives with respect to coordinates of the base S^{n-1} are subsummed in the Laplace-Beltrami operator $\Delta_{S^{n-1}}$ which has especially nice spectral properties that enable the use of a spectral resolution in (1.13). Furthermore, the direct inversion (1.13) requires that all coefficients a_j , $j = 0, 1, 2$, in the representation (1.6), are constants, which is obviously the case for the Laplace operator. However, this will not be the case in general and we have to rely on an asymptotic parametrix construction [8], which is considerably more involved.

2 A generalized approach for elliptic partial differential operators in the pseudo-differential algebra

The Laplace operator discussed in Section 1.1 motivates a generalization of our approach to elliptic second order differential operators on open stretched cones $\mathcal{C}^\wedge(X)$, with closed compact, $n - 1$ -dimensional smooth base manifold X , of the following form

$$\tilde{\mathcal{A}} = r^{-2} \left[\sum_{j=0}^2 a_j(r) \left(-r \frac{\partial}{\partial r} \right)^j + b(r) \Lambda_X \right] \quad (2.1)$$

and real analytic coefficients $a_j, b \in C^\infty(\bar{\mathbb{R}}^+)^2$. It should be mentioned, that despite the analyticity of the coefficients, (2.1) might be singular, due to prefactor r^{-2} in front of the differential operator. Furthermore, we assume that Λ_X is an elliptic, essentially self-adjoint, second order differential operator, semibounded from below, on the base X . Furthermore, we assume that Λ_X has a pure point spectrum $\lambda_0, \lambda_1, \lambda_2 \dots$, with lowest eigenvalue $\lambda_0 = 0$ and a corresponding constant eigenfunction, such that the corresponding eigenfunctions form a complete basis in $L_2(X)$.

For differential operators (2.1), we want to derive an explicit expression of the kernel of the parametrix in terms of a generalized Laplace expansion of the form

$$K(r, \phi | \tilde{r}, \tilde{\phi}) = \sum_{\ell=0}^{\infty} k_\ell(r, \tilde{r}, \lambda_\ell) p_\ell(\phi, \tilde{\phi}), \quad (2.2)$$

where $p_\ell(\phi | \tilde{\phi})$ denotes the integral kernel of the projection operator P_ℓ , $\ell = 0, 1, 2, \dots$, onto the $L_2(X)$ subspace spanned by the eigenfunctions of Λ_X for the eigenvalue λ_ℓ , i.e.,

$$p_\ell(\phi, \tilde{\phi}) = \sum_{m=1}^{m_\ell} u_{\ell,m}(\phi) \bar{u}_{\ell,m}(\tilde{\phi}), \quad \text{with } \Lambda_X u_{\ell,m} = \lambda_\ell u_{\ell,m}, \text{ for } m = 1, \dots, m_\ell.$$

In order to deal with r dependent coefficients a_j, b , $j = 0, 1, 2$, it becomes necessary to perform an asymptotic parametrix construction, introduced in Ref. [8] from which we take notations and definitions in the following. The differential operator (2.1) can be represented in the cone algebra by the Mellin type pseudo-differential operator

$$\tilde{\mathcal{A}} = r^{-2} \text{op}_M^{\gamma - \frac{n-1}{2}}(h(r, w, \Lambda_X))$$

where

$$h(r, w, \Lambda_X) := \sum_{j=0}^2 a_j(r) w^j + b(r) \Lambda_X$$

belongs to $C^\infty(\bar{\mathbb{R}}^+, M_{\mathcal{O}}^2(X))$, cf. [20] for further details.

2.1 Construction of an asymptotic parametrix

Given power series representations for the coefficients

$$a_j(r) = \sum_{p=0}^{\infty} a_j^{(p)} r^p, \quad j = 0, 1, 2 \text{ and } b(r) = \sum_{p=0}^{\infty} b^{(p)} r^p$$

²In our examples, we consider only the case where these coefficients are polynomials in r of finite order.

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$$h(r, w, \Lambda_X) = \sum_{i=0}^{\infty} r^i h_i(w, \Lambda_X)$$

where

$$h_i(w, \Lambda_X) = \sum_{j=0}^2 a_j^{(i)} w^j + b^{(i)} \Lambda_X$$

denotes the asymptotic symbols of the differential operator (2.1). From the ellipticity condition EC2 of the cone algebra, cf. Appendix D, we have the constraints $a_2^{(0)}, b^{(0)} \neq 0$ and that these coefficients must be of opposite sign. In the following we take $\gamma \in \mathbb{R}$ such that the ellipticity condition EC3 with respect to the conormal symbol, which in our case equals h_0 , is satisfied, i.e., h_0 defines isomorphisms $h_0(w, \Lambda_X) : H^s(X) \rightarrow H^{s-2}(X)$ for any $s \in \mathbb{R}$ and $\Re w = \frac{n}{2} - \gamma$. With respect to the coefficients, this condition is equivalent to

$$a_2^{(0)} w^2 + a_1^{(0)} w + a_0^{(0)} + b^{(0)} \lambda_\ell \neq 0, \quad \ell = 0, 1, 2, \dots$$

for all $\Re w = \frac{n}{2} - \gamma$. For its corresponding asymptotic parametrix, we take

$$r^2 \operatorname{op}_M^{\gamma - \frac{n+3}{2}}(h^{(-1)}(r, w, \Lambda_X))$$

with formal power series ansatz

$$h^{(-1)}(r, w, \Lambda_X) := \sum_{j=0}^{\infty} r^j h_j^{(-1)}(w, \Lambda_X).$$

for its symbol.

Remark 1. *For notational simplicity, we occasionally represent symbols of pseudo-differential operators by formal power series. Notwithstanding the fact that in the standard calculus such expressions are avoided due to its asymptotic character and lack of convergence in general. In our applications considered in the present work, however, we always perform calculations of parametrix symbols to infinite order and prove convergence of the derived fundamental solutions or Green's functions in hindsight.*

For notational simplicity we suppress the Λ_X dependence of the symbols for a while. As a first step, we commute the r^{-2} factor to the right

$$\begin{aligned} P\tilde{A} &= r^2 \operatorname{op}_M^{\gamma - \frac{n+3}{2}}(h^{(-1)}(r, w)) r^{-2} \operatorname{op}_M^{\gamma - \frac{n-1}{2}}(h(r, w)) \\ &= \operatorname{op}_M^{\gamma - \frac{n-1}{2}}(h^{(-1)}(r, w+2)) \operatorname{op}_M^{\gamma - \frac{n-1}{2}}(h(r, w)). \end{aligned} \quad (2.3)$$

Inserting the power series of the symbols and shifting all powers of r to the right, we obtain

$$\begin{aligned} P\tilde{A} &= \left[\sum_{j=0}^{\infty} r^j \operatorname{op}_M^{\gamma - \frac{n-1}{2}}(h_j^{(-1)}(w+2)) \right] \left[\sum_{i=0}^N r^i \operatorname{op}_M^{\gamma - \frac{n-1}{2}}(h_i(w)) \right] \\ &= \sum_{j,i=0}^{\infty} r^{j+i} \operatorname{op}_M^{\gamma - \frac{n-1}{2} - i}(h_j^{(-1)}(w+2-i)) \operatorname{op}_M^{\gamma - \frac{n-1}{2}}(h_i(w)) \\ &= \sum_{j,i=0}^{\infty} r^{j+i} \operatorname{op}_M^{\gamma - \frac{n-1}{2}}(h_j^{(-1)}(w+2-i)) \operatorname{op}_M^{\gamma - \frac{n-1}{2}}(h_i(w)) \mod G \end{aligned}$$

where the last step is modulo Green's operators, which will be discussed in detail below. After the manipulations, we can now define the symbols of the parametrix in a recursive manner. For the symbol $h_0^{(-1)}$ we get the equation

$$1 = \text{op}_M^{\gamma - \frac{n-1}{2}}(h_0^{(-1)}(w+2)) \text{op}_M^{\gamma - \frac{n-1}{2}}(h_0(w)) = \text{op}_M^{\gamma - \frac{n-1}{2}}(h_0^{(-1)}(w+2)h_0(w))$$

and therefore

$$\left(h_0^{(-1)}(w+2)\right)h_0(w) = 1 \longrightarrow h_0^{(-1)}(w) = h_0^{-1}(w-2). \quad (2.4)$$

The equations of the symbols $h_j^{(-1)}$, for $j \geq 1$, are

$$0 = \text{op}_M^{\gamma - \frac{n-1}{2}}(h_j^{(-1)}(w+2)) \text{op}_M^{\gamma - \frac{n-1}{2}}(h_0(w)) + \sum_{i=1}^j \text{op}_M^{\gamma - \frac{n-1}{2}}(h_{j-i}^{(-1)}(w+2-i)) \text{op}_M^{\gamma - \frac{n-1}{2}}(h_i(w)),$$

which gives

$$0 = (h_j^{(-1)}(w+2))h_0(w) + \sum_{i=1}^j (h_{j-i}^{(-1)}(w+2-i))h_i(w),$$

and

$$\begin{aligned} h_j^{(-1)}(w) &= - \left(\sum_{i=1}^j (h_{j-i}^{(-1)}(w-i)) (h_i(w-2)) \right) (h_0^{-1}(w-2)) \\ &= - \left(\sum_{i=1}^j (h_{j-i}^{(-1)}(w-i)) (h_i(w-2)) \right) h_0^{(-1)}(w) \end{aligned} \quad (2.5)$$

respectively. In order to derive integral kernels, we have to know the poles of the meromorphic operator valued symbols $h_j^{(-1)}(w)$. It can be seen from (2.5) that the poles are recursively defined such that $h_j^{(-1)}(w)$ has meromorphic terms of the form

$$\begin{aligned} j=0 : & \quad h_0^{(-1)}(w) \\ j=1 : & \quad (h_0^{(-1)}(w-1))h_0^{(-1)}(w) \\ j=2 : & \quad (h_0^{(-1)}(w-2))(h_0^{(-1)}(w-1))h_0^{(-1)}(w), \quad (h_0^{(-1)}(w-2))h_0^{(-1)}(w) \\ j=3 : & \quad (h_0^{(-1)}(w-3))(h_0^{(-1)}(w-2))(h_0^{(-1)}(w-1))h_0^{(-1)}(w), \quad (h_0^{(-1)}(w-3))(h_0^{(-1)}(w-1))h_0^{(-1)}(w), \\ & \quad (h_0^{(-1)}(w-3))(h_0^{(-1)}(w-2))h_0^{(-1)}(w), \quad (h_0^{(-1)}(w-3))h_0^{(-1)}(w) \\ & \quad \vdots \end{aligned} \quad (2.6)$$

After re-emerging of Λ_X in the symbols of the parametrix, we can perform spectral resolutions

$$\begin{aligned} h_0^{(-1)}(w, \Lambda_X) &= \sum_{\ell=0}^{\infty} h_0^{(-1)}(w, \lambda_\ell) P_\ell, \quad h_i(w, \Lambda_X) = \sum_{\ell=0}^{\infty} h_i(w, \lambda_\ell) P_\ell \\ h_0^{(-1)}(w, \lambda_\ell) &= \frac{1}{\sum_{j=0}^2 a_j^{(0)}(w-2)^j + b^{(0)}\lambda_\ell}, \quad h_i(w, \lambda_\ell) := \sum_{j=0}^2 a_j^{(i)} w^j + b^{(i)}\lambda_\ell \end{aligned}$$

and use the orthogonality of the projection operators, i.e. $P_\ell P_{\bar{\ell}} = \delta_{\ell, \bar{\ell}} P_\ell$, to get the corresponding spectral resolution of the Mellin symbols of the parametrix

$$h_j^{(-1)}(w, \Lambda_X) = \sum_{\ell=0}^{\infty} h_j^{(-1)}(w, \lambda_\ell) P_\ell$$

with

$$h_j^{(-1)}(w, \lambda_\ell) = - \left(\sum_{i=1}^j (h_{j-i}^{(-1)}(w-i, \lambda_\ell)) (h_i(w-2, \lambda_\ell)) \right) h_0^{(-1)}(w, \lambda_\ell)$$

where the latter symbols are defined in a recursive manner. According to our previous discussion, the poles of $h_j^{(-1)}(w, \lambda_\ell)$ are given by the poles of the shifted symbols $h_0^{(-1)}(w-m)$, for $m = 0, 1, \dots, j$. The symbol $h_0^{(-1)}(w)$ has poles at

$$w(\lambda_\ell) = 2 - \frac{a_1^{(0)}}{2a_2^{(0)}} \pm \sqrt{\left(\frac{a_1^{(0)}}{2a_2^{(0)}} \right)^2 - \frac{a_0^{(0)}}{a_2^{(0)}} - \frac{b^{(0)}}{a_2^{(0)}} \lambda_\ell}.$$

In order to simplify our discussion, we want to restrict in the following to two different types of differential operators.

Definition 3. Let $\tilde{A}$ be an elliptic differential operator in $\text{Diff}_{\deg}^2(\mathcal{C}^\wedge(X))$ of the form (2.1), where we assume w.l.o.g. $a_2^{(0)} = 1$ for notational simplicity. A differential operator is referred as type-A if $(a_1^{(0)})^2 - 4a_0^{(0)} > 0$, $b^{(0)} < 0$, and as type-B if $(a_1^{(0)})^2 - 4a_0^{(0)} = 0$, $b^{(0)} < 0$.

Remark 2. It follows from the ellipticity condition EC2 of the cone algebra, cf. Appendix D, that $a_2^{(0)} b^{(0)} < 0$ and therefore our choice $a_2^{(0)} = 1$ implies $b^{(0)} < 0$.

Remark 3. According to our definition, the symbol $h_0^{(-1)}$ has two poles of order 1 if it corresponds to a type-A differential operator and a single pole of order 2 if it corresponds to a type-B differential operator.

Example 2. The Laplacian Δ_n in $\mathbb{R}^n$ is of type-A for $n \geq 3$ and of type-B for $n = 2$.

In the present work, we want to focus on type-A differential operators and leave type-B differential operators for a subsequent publication.

2.1.1 Parametrices for type-A differential operators

Let us consider asymptotic parametrices for type-A differential operators. For notational simplicity, we introduce the λ_ℓ dependent parameter

$$\Delta w(\lambda_\ell) := \sqrt{\left(\frac{a_1^{(0)}}{2} \right)^2 - a_0^{(0)} - b^{(0)} \lambda_\ell}.$$

Let us first consider the operator (2.1) with constant coefficients, which corresponds to the 0-th order term in the expression of $\tilde{\mathcal{A}}$ with respect to powers of r .

Performing a similar calculation as in Section 1.1, with $u \in \tilde{\mathcal{D}}(\mathcal{C}^\wedge)$, we get:

$$\begin{aligned}
\mathcal{P}_0 u(r, \phi) &= - \sum_{\ell=0}^{\infty} \int_X \int_0^r \frac{\tilde{r}^{w_{1,\ell}-n}}{r^{w_{1,\ell}-2}} \frac{p_\ell(\phi|\tilde{\phi})}{2\Delta w(\lambda_\ell)} u(\tilde{r}, \tilde{\phi}) \tilde{r}^{n-1} d\tilde{r} \mu_{n-1}(\tilde{\phi}) d\tilde{\phi} \\
&\quad - \sum_{\ell=0}^{\infty} \int_X \int_r^\infty \frac{\tilde{r}^{w_{2,\ell}-n}}{r^{w_{2,\ell}-2}} \frac{p_\ell(\phi|\tilde{\phi})}{2\Delta w(\lambda_\ell)} u(\tilde{r}, \tilde{\phi}) \tilde{r}^{n-1} d\tilde{r} \mu_{n-1}(\tilde{\phi}) d\tilde{\phi} \\
&= -r^{a_1^{(0)}/2} \sum_{\ell=0}^{\infty} \int_X \int_0^\infty \tilde{r}^{2-n-a_1^{(0)}/2} \left(\frac{r \leq}{r >} \right)^{\Delta w(\lambda_\ell)} \frac{p_\ell(\phi|\tilde{\phi})}{2\Delta w(\lambda_\ell)} u(\tilde{r}, \tilde{\phi}) \tilde{r}^{n-1} d\tilde{r} \mu_{n-1}(\tilde{\phi}) d\tilde{\phi},
\end{aligned}$$

with

$$w_{1,\ell} := 2 - \frac{a_1^{(0)}}{2} + \Delta w(\lambda_\ell), \quad w_{2,\ell} := 2 - \frac{a_1^{(0)}}{2} - \Delta w(\lambda_\ell),$$

where we choose an integration contour $\Gamma_{\frac{n+4}{2}-\gamma}$ between the two poles, i.e.,

$$\frac{n}{2} + \frac{a_1^{(0)}}{2} - \Delta w(\lambda_\ell) < \gamma < \frac{n}{2} + \frac{a_1^{(0)}}{2} + \Delta w(\lambda_\ell).$$

This particular choice is motivated by our considerations for the Laplace operator in Section 1.1. Eventually our choice can be justified by explicit constructions of fundamental solutions and Green's functions from the parametrices. In the following, we will provide evidence by considering selected examples for type-A differential operators.

The parametrix $\mathcal{P}_0$ can be represented by an integral operator

$$\mathcal{P}_0 u(r, \phi) = \int_X \int_0^\infty K_0(r, \phi|\tilde{r}, \tilde{\phi}) u(\tilde{r}, \tilde{\phi}) \tilde{r}^{n-1} d\tilde{r} \mu_{n-1}(\tilde{\phi}) d\tilde{\phi}$$

with kernel function

$$K_0(r, \phi|\tilde{r}, \tilde{\phi}) = -r^{a_1^{(0)}/2} \tilde{r}^{2-n-a_1^{(0)}/2} \sum_{\ell=0}^{\infty} \left(\frac{r \leq}{r >} \right)^{\Delta w(\lambda_\ell)} \frac{p_\ell(\phi|\tilde{\phi})}{2\Delta w(\lambda_\ell)}$$

Lemma 1. *Let $\tilde{\mathcal{A}}_0$ be an elliptic type-A differential operator in $\mathcal{C}^\wedge(X)$, i.e.,*

$$\tilde{\mathcal{A}}_0 = r^{-2} \left[\left(-r \frac{\partial}{\partial r} \right)^2 + a_1^{(0)} \left(-r \frac{\partial}{\partial r} \right) + a_0^{(0)} + b^{(0)} \Lambda_X \right], \quad (2.7)$$

with coefficients satisfying

$$a_0^{(0)} = -a_1^{(0)}(n-2) - (n-2)^2. \quad (2.8)$$

The distribution $\text{Pf. } k_0 \in \mathcal{D}'(\mathbb{R}^n)$, with

$$k_0(r) := \lim_{\tilde{r} \rightarrow 0} K_0(r, \phi|\tilde{r}, \tilde{\phi}) = -r^{2-n} \frac{p_0}{a_1^{(0)} + 2(n-2)} \quad (2.9)$$

is a fundamental solution of the corresponding differential operator $\mathcal{A}_0$ in $\mathbb{R}^n$, i.e.,

$$\mathcal{A}_0 \text{Pf. } k_0 = \delta.$$

Proof. Under the first assumption in (2.8) the limit (2.9) follows immediately.

For any test function $g \in \mathcal{D}(\mathbb{R}^n)$ let $\tilde{g} := \varphi^* g$ be its corresponding counterpart in $\tilde{\mathcal{D}}(C^\wedge)$. Due to the spherical symmetry of k_0 , we get³

$$\begin{aligned} & \int_{\mathbb{R}^n} g(x) \mathcal{A}_0 \text{Pf. } k_0(x) dx \\ &= \int_0^\infty r^{n-1} \frac{1}{r^2} \left[\left((n-2) + r \frac{\partial}{\partial r} \right)^2 \tilde{g}_0(r) + a_1^{(0)} \left((n-2) + r \frac{\partial}{\partial r} \right) \tilde{g}_0(r) + a_0^{(0)} \tilde{g}_0(r) \right] \frac{k_0(r)}{p_0} dr \end{aligned}$$

where the right hand side only depends on the projection $\tilde{g}_0(r) := P_0 \tilde{g}$. Calculating the integrals

$$\begin{aligned} & - \int_0^\infty r^{n-1} \frac{1}{r^2} \left(r \frac{\partial}{\partial r} \tilde{g}_0(r) \right) r^{2-n} \frac{1}{2\Delta w(\lambda_0)} dr \\ &= - \int_0^\infty \partial_r \tilde{g}_0(r) \frac{1}{2\Delta w(\lambda_0)} dr \\ &= \lim_{r \rightarrow 0} \frac{\tilde{g}_0(r)}{2\Delta w(\lambda_0)} \\ &= \frac{g(0)}{2\Delta w(\lambda_0)} \end{aligned}$$

and

$$\begin{aligned} & - \int_0^\infty r^{n-1} \frac{1}{r^2} \left(r \frac{\partial}{\partial r} \right) \left[r \frac{\partial}{\partial r} \tilde{g}_0(r) \right] r^{2-n} \frac{1}{2\Delta w(\lambda_0)} dr \\ &= - \int_0^\infty \partial_r (r \partial_r \tilde{g}_0(r)) \frac{1}{2\Delta w(\lambda_0)} dr \\ &= 0, \end{aligned}$$

we finally obtain

$$\int_{\mathbb{R}^n} g(x) \mathcal{A}_0 \text{Pf. } k_0(x) dx = g(0).$$

□

It remains to consider the higher order terms in the asymptotic parametrix construction for type-A differential operators.

Proposition 1. *Parametrix symbols $h_j^{(-1)}(w, \lambda_\ell)$ of higher order, i.e. $j = 1, 2, \dots$, which correspond to type-A differential operators have generically poles of order 1, however in particular cases second order poles may show up. For a symbol $h_j^{(-1)}(w, \lambda_\ell)$ poles at most of order 2 can appear only if*

$$2\Delta w(\lambda_\ell) \in \{1, 2, \dots, j\}. \quad (2.10)$$

³Here and in the following, we make a slight abuse of notation concerning partial differential operators acting on pseudofunctions. Generally it yields a distribution which is not regular any more, e.g., $\mathcal{A}_0 \text{Pf. } k_0$ is not a regular distribution. However, we make use of the pervasive notation which treats e.g. Dirac's delta distribution like a function, i.e., $\delta(g) = \int \delta(x)g(x)dx$.

If this is the case, then $h_j^{(-1)}(w, \lambda_\ell)$ potentially has $q_\ell := j - k_{\min}(\lambda_\ell) + 1$ poles of order 2 at

$$w_m(\lambda_\ell) = 2 - \frac{a_1^{(0)}}{2} + m + \Delta w(\lambda_\ell), \quad m = 0, 1, \dots, j - k_{\min}(\lambda_\ell),$$

with $k_{\min}(\lambda_\ell) = 2\Delta w(\lambda_\ell)$.

Proof. The symbols $h_j^{(-1)}(w, \lambda_\ell)$ given by the recursive formula (2.5) have zeros in the denominator, cf. (2.6), at

$$w_{1,m}(\lambda_\ell) = 2 - \frac{a_1^{(0)}}{2} + m + \Delta w(\lambda_\ell), \quad w_{2,k}(\lambda_\ell) = 2 - \frac{a_1^{(0)}}{2} + k - \Delta w(\lambda_\ell), \quad m, k = 0, \dots, j.$$

Therefore a zero of order 2 can appear only if

$$k - m = 2\Delta w(\lambda_\ell),$$

which leads to a pole of order 2 at $w_0 := w_{1,j} = w_{1,k}$ provided $w - w_0$ does not divide the corresponding nominator. So let $k_{\min}(\lambda_l) := 2\Delta w(\lambda_\ell)$, then $h_j^{(-1)}(w, \lambda_\ell)$ potentially has $q_\ell = j - k_{\min}(\lambda_l) + 1$ poles of order 2 at

$$w_m = 2 - \frac{a_1^{(0)}}{2} + m + \Delta w(\lambda_\ell), \quad m = 0, 1, \dots, j - k_{\min}(\lambda_\ell).$$

□

Let us assume that the symbol $h_j^{(-1)}(w, \lambda_\ell)$ has $2j + 2 - 2q_\ell$, $q_\ell \geq 0$, simple poles, which we arrange for each λ_ℓ in descending order, i.e.,

$$w_1(\lambda_\ell) > w_2(\lambda_\ell) > \dots > w_{2p+2-2q_\ell}(\lambda_\ell)$$

where $w_1(\lambda_\ell), \dots, w_{j_\ell}(\lambda_\ell)$ lies right and $w_{j_\ell+1}(\lambda_\ell), \dots, w_{2p+2}(\lambda_\ell)$ left of the integration contour. The remaining poles of order 2, denoted by $w_k^{(2)}(\lambda_\ell)$, $k = 1, \dots, q_\ell$, are accordingly on the right side of the integration contour and are as well arranged in descending order, i.e.,

$$w_1^{(2)}(\lambda_\ell) > w_2^{(2)}(\lambda_\ell) > \dots > w_{q_\ell}^{(2)}(\lambda_\ell).$$

The residue of such a poles is given by

$$\begin{aligned} \text{Res}_{w_k^{(2)}(\lambda_\ell)} \left(\frac{r}{\tilde{r}} \right)^{-w} h_j^{(-1)}(w) &= \lim_{w \rightarrow w_k^{(2)}} \frac{d}{dw} \left[(w - w_k^{(2)}(\lambda_\ell))^2 \left(\frac{r}{\tilde{r}} \right)^{-w} h_j^{(-1)}(w) \right] \\ &= -\ln \left(\frac{r}{\tilde{r}} \right) \left(\frac{\tilde{r}}{r} \right)^{w_k^{(2)}} \text{Res}_{w_k^{(2)}(\lambda_\ell)}^{(1)} (h_j^{(-1)}(\cdot, \lambda_\ell)) + \left(\frac{\tilde{r}}{r} \right)^{w_k^{(2)}} \text{Res}_{w_k^{(2)}(\lambda_\ell)}^{(2)} (h_j^{(-1)}(\cdot, \lambda_\ell)) \end{aligned}$$

with

$$\begin{aligned} \text{Res}_{w_k^{(2)}(\lambda_\ell)}^{(1)} (h_j^{(-1)}(\cdot, \lambda_\ell)) &:= \lim_{w \rightarrow w_k^{(2)}(\lambda_\ell)} \left[(w - w_k^{(2)}(\lambda_\ell))^2 h_j^{(-1)}(w) \right], \\ \text{Res}_{w_k^{(2)}(\lambda_\ell)}^{(2)} (h_j^{(-1)}(\cdot, \lambda_\ell)) &:= \lim_{w \rightarrow w_k^{(2)}(\lambda_\ell)} \frac{d}{dw} \left[(w - w_k^{(2)}(\lambda_\ell))^2 h_j^{(-1)}(w) \right]. \end{aligned}$$

Putting things together, we obtain with $u \in \tilde{\mathcal{D}}(\mathcal{C}^\wedge)$

$$\begin{aligned}
\mathcal{P}_j u(r, \phi) &= - \sum_{\ell=0}^{\infty} \sum_{k=1}^{j_\ell} \int_X \int_0^r \frac{\tilde{r}^{w_k(\lambda_\ell)-n}}{r^{w_k(\lambda_\ell)-2}} (\text{Res}_{w_k(\lambda_\ell)} h_j^{(-1)}(\cdot, \lambda_\ell)) p_\ell(\phi|\tilde{\phi}) u(\tilde{r}, \tilde{\phi}) \tilde{r}^{n-1} d\tilde{r} \mu_{n-1}(\tilde{\phi}) d\tilde{\phi} \\
&+ \sum_{\ell=0}^{\infty} \sum_{k=j_\ell+1}^{2p+2-2q_\ell} \int_X \int_r^\infty \frac{\tilde{r}^{w_k(\lambda_\ell)-n}}{r^{w_k(\lambda_\ell)-2}} (\text{Res}_{w_k(\lambda_\ell)} h_j^{(-1)}(\cdot, \lambda_\ell)) p_\ell(\phi|\tilde{\phi}) u(\tilde{r}, \tilde{\phi}) \tilde{r}^{n-1} d\tilde{r} \mu_{n-1}(\tilde{\phi}) d\tilde{\phi} \\
&+ \sum_{\ell=0}^{\infty} \sum_{k=1}^{q_\ell} \int_X \int_0^r \frac{\tilde{r}^{w_k^{(2)}(\lambda_\ell)-n}}{r^{w_k^{(2)}(\lambda_\ell)-2}} \left[\ln\left(\frac{r}{\tilde{r}}\right) \text{Res}_{w_k^{(2)}(\lambda_\ell)}^{(1)} h_j^{(-1)}(\cdot, \lambda_\ell) \right. \\
&\quad \left. - \text{Res}_{w_k^{(2)}(\lambda_\ell)}^{(2)} h_j^{(-1)}(\cdot, \lambda_\ell) \right] p_\ell(\phi|\tilde{\phi}) u(\tilde{r}, \tilde{\phi}) \tilde{r}^{n-1} d\tilde{r} \mu_{n-1}(\tilde{\phi}) d\tilde{\phi}
\end{aligned}$$

Let us summarize our calculations in the following lemma

Lemma 2. *For a type-A differential operator $\tilde{\mathcal{A}}$, the asymptotic parametrix given by the series*

$$\mathcal{P} = r^2 \sum_{j=0}^{\infty} r^j \text{op}_M^{\gamma - \frac{n+3}{2}} (h_j^{(-1)}(w, \Lambda_X))$$

can be represented, for $u \in \tilde{\mathcal{D}}(\mathcal{C}^\wedge)$, by an integral operator

$$\mathcal{P} u(r, \phi) = \sum_{j=0}^{\infty} r^j \int_X \int_0^\infty K_j(r, \phi|\tilde{r}, \tilde{\phi}) u(\tilde{r}, \tilde{\phi}) \tilde{r}^{n-1} d\tilde{r} \mu_{n-1}(\tilde{\phi}) d\tilde{\phi}$$

with kernel functions

$$\begin{aligned}
K_j(r, \phi|\tilde{r}, \tilde{\phi}) &= -H(r - \tilde{r}) \sum_{\ell=0}^{\infty} \sum_{k=1}^{j_\ell} r^{2-w_k(\lambda_\ell)} \tilde{r}^{w_k(\lambda_\ell)-n} (\text{Res}_{w_k(\lambda_\ell)} h_j^{(-1)}(\cdot, \lambda_\ell)) p_\ell(\phi|\tilde{\phi}) \\
&+ H(r - \tilde{r}) \sum_{\ell=0}^{\infty} \sum_{k=1}^{q_\ell} r^{2-w_k^{(2)}(\lambda_\ell)} \tilde{r}^{w_k^{(2)}(\lambda_\ell)-n} \left[\ln\left(\frac{r}{\tilde{r}}\right) \text{Res}_{w_k^{(2)}(\lambda_\ell)}^{(1)} h_j^{(-1)}(\cdot, \lambda_\ell) \right. \\
&\quad \left. - \text{Res}_{w_k^{(2)}(\lambda_\ell)}^{(2)} h_j^{(-1)}(\cdot, \lambda_\ell) \right] p_\ell(\phi|\tilde{\phi}) \\
&+ H(\tilde{r} - r) \sum_{\ell=0}^{\infty} \sum_{k=j_\ell+1}^{2p+2-2q_\ell} r^{2-w_k(\lambda_\ell)} \tilde{r}^{w_k(\lambda_\ell)-n} (\text{Res}_{w_k(\lambda_\ell)} h_j^{(-1)}(\cdot, \lambda_\ell)) p_\ell(\phi|\tilde{\phi}).
\end{aligned} \tag{2.11}$$

Here H denotes the Heaviside function, i.e., $H(r - \tilde{r}) = \begin{cases} 0, & r < \tilde{r} \\ 1, & r > \tilde{r} \end{cases}$.

Lemma 1 shows that for $a_0^{(0)} = -a_1^{(0)}(n-2) - (n-2)^2$, the limit $\tilde{r} \rightarrow 0$ of K_0 yields a fundamental solution of $\mathcal{A}_0$. Under these conditions, we get $2\Delta w(\lambda_0) = a_1^{(0)} + 2(n-2)$, which can lead, according to Proposition 1, for integer $a_1^{(0)}$ to poles of order 2 in the asymptotic parametrix construction if

(2.10) is satisfied. In such a case, due to the presence of logarithmic terms, it is not possible to take the limit $\tilde{r} \rightarrow 0$ of K_j for $j \geq 2\Delta w(\lambda_0)$. It should be emphasized, that this problem only concerns λ_0 , because for higher eigenvalues, the prefactors $\tilde{r}^{w_k(\lambda_\ell)-n}$, $\ell = 1, 2, \dots$, have positive exponents and the terms $\tilde{r}^{w_k(\lambda_\ell)-n} \ln(\tilde{r})$ vanish in the limit $\tilde{r} \rightarrow 0$. Furthermore it might happen, that poles are located on the real axis between the integration contour $\Gamma_{\frac{n+4}{2}-\gamma}$ and the pole $w_{j_n}(\lambda_0) = n$. According to our construction, we only assume that $\Gamma_{\frac{n+4}{2}-\gamma}$ cuts the real axis between the poles $w_{j_{n-1}}(\lambda_0) = 4 - a_1^{(0)} - n$ and $w_{j_n}(\lambda_0) = n$, so depending on the choice of γ such cases might occur.

For $r > \tilde{r}$, we decompose the kernel according to

$$K_j(r, \phi | \tilde{r}, \tilde{\phi}) = K_j^{(0)}(r, \phi | \tilde{r}, \tilde{\phi}) + K_j^{(1)}(r, \phi | \tilde{r}, \tilde{\phi}) \quad (r > \tilde{r})$$

with

$$\begin{aligned} K_j^{(0)}(r, \phi | \tilde{r}, \tilde{\phi}) &:= - \sum_{\ell=0}^{\infty} \sum_{k=1}^{j_\ell} H_0(w_k(\lambda_\ell) - n) r^{2-w_k(\lambda_\ell)} \tilde{r}^{w_k(\lambda_\ell)-n} (\text{Res}_{w_k(\lambda_\ell)} h_j^{(-1)}(\cdot, \lambda_\ell)) p_\ell(\phi | \tilde{\phi}) \\ &+ \sum_{\ell=0}^{\infty} \sum_{k=1}^{q_\ell} r^{2-w_k(\lambda_\ell)} \tilde{r}^{w_k(\lambda_\ell)-n} \left[\ln(r) \text{Res}_{w_k^{(2)}(\lambda_\ell)}^{(1)} h_j^{(-1)}(\cdot, \lambda_\ell) \right. \\ &\quad \left. - \text{Res}_{w_k^{(2)}(\lambda_\ell)}^{(2)} h_j^{(-1)}(\cdot, \lambda_\ell) \right] p_\ell(\phi | \tilde{\phi}) \end{aligned}$$

$$\begin{aligned} K_j^{(1)}(r, \phi | \tilde{r}, \tilde{\phi}) &:= - \sum_{\ell=0}^{\infty} \sum_{k=1}^{j_\ell} H_1(n - w_k(\lambda_\ell)) r^{2-w_k(\lambda_\ell)} \tilde{r}^{w_k(\lambda_\ell)-n} (\text{Res}_{w_k(\lambda_\ell)} h_j^{(-1)}(\cdot, \lambda_\ell)) p_\ell(\phi | \tilde{\phi}) \\ &- \sum_{\ell=0}^{\infty} \sum_{k=1}^{q_\ell} r^{2-w_k(\lambda_\ell)} \tilde{r}^{w_k(\lambda_\ell)-n} \ln(\tilde{r}) \text{Res}_{w_k^{(2)}(\lambda_\ell)}^{(1)} h_j^{(-1)}(\cdot, \lambda_\ell) p_\ell(\phi | \tilde{\phi}), \end{aligned}$$

where the Heavyside functions are defined at 0 by $H_0(0) = 1$, $H_1(0) = 0$, respectively. Taking the limit $\tilde{r} \rightarrow 0$ of the average of $K_j^{(0)}$, we obtain

$$\begin{aligned} k_j(r) &:= \lim_{\tilde{r} \rightarrow 0} \frac{1}{A_X} \int_X K_j^{(0)}(r, \phi | \tilde{r}, \tilde{\phi}) \mu_{n-1}(\tilde{\phi}) d\tilde{\phi} = -r^{2-n} \sum_{k=1}^{j_0} \delta(w_k(\lambda_0) - n) (\text{Res}_{w_k(\lambda_0)} h_j^{(-1)}(\cdot, \lambda_0)) p_0 \\ &+ r^{2-n} \sum_{k=1}^{q_0} \delta(w_k^{(2)}(\lambda_0) - n) \left[\ln(r) \text{Res}_{w_k^{(2)}(\lambda_0)}^{(1)} h_j^{(-1)}(\cdot, \lambda_0) \right. \\ &\quad \left. - \text{Res}_{w_k^{(2)}(\lambda_0)}^{(2)} h_j^{(-1)}(\cdot, \lambda_0) \right] p_0, \end{aligned} \tag{2.12}$$

where A_X denotes the surface area of X .

Remark 4. *Logarithmic terms can only appear for $j \geq 2\Delta w(\lambda_0)$. For each order j of the asymptotic parametrix construction, there are only a finite number of non vanishing terms in the sum over ℓ . Furthermore, there is at most only one nonvanishing term on the right hand side of (2.12).*

3 Shifted Laplace operators

As a first example for the asymptotic parametrix construction, we consider the shifted Laplace operator $\tilde{\Delta}_n - \kappa^2$ in dimension $n \geq 3$ with $\tilde{\Delta}_n$ given by (1.10). The shifted Laplace operator is of type-A with base $X = S^{n-1}$ and $\Lambda_X = -\Delta_{S^{n-1}}$. Its Mellin symbol has the form

$$h(w) = h_0(w) + r^2 h_2(w)$$

with h_0 and h_2 given by (1.12) and $-\kappa^2$, respectively. Application of the recursion formula (2.5) yields

$$h_{2m}^{(-1)}(w) = \kappa^{2m} \prod_{j=0}^m h_0^{(-1)}(w - 2j) \quad \text{and} \quad h_{2m+1}^{(-1)} = 0 \quad \text{for } m = 0, 1, \dots, \quad (3.1)$$

where $h_0^{(-1)}$ is given by (1.13).

3.1 Fundamental solutions from the parametrix

To exemplify our approach, we only consider the cases $n = 3, 4$ in the following proposition. However it is clear that analogous calculations can be performed in any dimension ≥ 3 .

Proposition 2. *For $\Delta_n - \kappa^2$, $n = 3, 4$, the asymptotic parametrix construction, with integration contour $\Gamma_{\frac{n+4}{2}-\gamma}$ taken for $2 - \frac{n}{2} < \gamma < 3 - \frac{n}{2}$, yields a fundamental solution*

$$u = \text{Pf. } k(r) \text{ with } k(r) := k_0(r) + rk_1(r) + r^2 k_2(r) \cdots,$$

where the functions k_p , $p = 0, 1, \dots$, are given by (2.12).

Proof. According to (3.1), the symbols $h_{2m,0}^{(-1)}$ have a pole at n if $2 + 2j$ or $n + 2j$ equals n for some $j \in \{0, 1, \dots, m\}$. Therefore for n odd, only poles of order 1 appear, whereas for n even, poles of order 1 and 2 show up. For an admissible γ , no poles appear on the strip between the integration contour and the pole at n .

Case 1: n odd

In each symbol $h_{2m}^{(-1)}$, there is only for $j = 0$ a pole of order 1 at n . Formulas (2.12) and (3.1) yield

$$k_{2m}(r) = -r^{2-n} \frac{(-1)^m \kappa^{2m} p_0}{2^m m!} \prod_{j=0}^m \frac{1}{n - 2j - 2}, \quad \text{with } p_0 = \frac{(n-2)!!}{2^{\frac{n+1}{2}} \pi^{\frac{n-1}{2}}}.$$

In particular, for $n = 3$ we get

$$k_{2m}(r) = -r^{-1} \frac{\kappa^{2m}}{4\pi(2m)!} \quad (3.2)$$

and

$$k(r) = \sum_{j=0}^{\infty} r^j k_j(r) = -\frac{1}{4\pi r} \sum_{m=0}^{\infty} \frac{(\kappa r)^{2m}}{(2m)!} = -\frac{1}{4\pi r} \cosh(\kappa r)$$

This can be compared with the standard fundamental solution, cf. [21],

$$u(x) = \text{Pf. } -\frac{1}{4\pi r} e^{-\kappa r}. \quad (3.3)$$

It can be easily seen, the difference between both fundamental solutions

$$\text{Pf.} - \frac{1}{4\pi r} (\cosh(\kappa r) - e^{-\kappa r}) = \text{Pf.} - \frac{1}{4\pi r} \sinh(\kappa r)$$

correspond to a smooth harmonic function in $\mathbb{R}^3$.

Case 2: n even

For $p < n - 2$, the symbols $h_{2m}^{(-1)}$ have a pole of order 1 and for $p \geq n$ a pole of order 2 at n . Let

$$Q_0(w) := \frac{1}{w-2} \quad \text{and} \quad Q_m(w) := \prod_{j=0, 2j+2 \neq n}^m \frac{1}{w-2j-2} \prod_{j=1}^m \frac{1}{w-2j-n} \quad \text{for } m \geq 1,$$

as well as

$$Q'_m(w) = \frac{dQ_m}{dw}(w) = -Q_m(w) \left[\sum_{j=0, 2j+2 \neq n}^m \frac{1}{w-2j-2} + \sum_{j=1}^m \frac{1}{w-2j-n} \right] \quad \text{for } m \geq 1.$$

With this, we get

$$\text{Res}_n h_{2m}^{(-1)} = \kappa^{2m} Q_m(n) \quad \text{for } 2m < n - 2,$$

and

$$\text{Res}_n^{(1)} h_{2m}^{(-1)} = \kappa^{2m} Q_m(n), \quad \text{Res}_n^{(2)} h_{2m}^{(-1)} = \kappa^{2m} Q'_m(n) \quad \text{for } 2m \geq n - 2.$$

Therefore, according to formulas (2.12) and (3.1),

$$k_{2m}(r) = -r^{2-n} \kappa^{2m} \begin{cases} Q_m(n) p_0 & \text{for } 2m < n - 2 \\ -(\ln(r) Q_m(n) - Q'_m(n)) p_0 & \text{for } 2m \geq n - 2 \end{cases}$$

Let us demonstrate the proposition by an explicit calculation for $n = 4$, where we have

$$Q_0(4) = \frac{1}{2}, \quad Q_m(4) = -\frac{1}{2^{2m} m! (m-1)!} \quad \text{for } m \geq 1$$

and

$$Q'_m(4) = -\frac{1}{2^{2m+1} m! (m-1)!} \left(-1 + \sum_{j=1}^{m-1} j^{-1} + \sum_{j=1}^m j^{-1} \right) \quad \text{for } m \geq 1.$$

Finally, with $p_0 = \frac{1}{2\pi^2}$, we get

$$k(r) = -\frac{1}{4\pi^2} r^{-2} - \sum_{m=1}^{\infty} r^{-2+2m} \frac{\kappa^{2m}}{2\pi^2} \left[\frac{1}{2^{2m} m! (m-1)!} \ln(r) - \frac{1}{2^{2m+1} m! (m-1)!} \left(-1 + \sum_{j=1}^{m-1} j^{-1} + \sum_{j=1}^m j^{-1} \right) \right]$$

and after some algebraic manipulations, we obtain

$$k(r) = -\frac{1}{4\pi^2} r^{-2} - \frac{1}{4\pi^2} (\kappa r)^{-1} I_1(\kappa r) \ln(r) + \frac{\kappa^2}{(4\pi)^2} \sum_{k=0}^{\infty} \frac{1}{2^{2k} k! (k+1)!} \left(-1 + \sum_{j=1}^k j^{-1} + \sum_{j=1}^{k+1} j^{-1} \right) (\kappa r)^{2k} \quad (3.4)$$

where we used, cf. [1][Eq. 9.6.10],

$$(\kappa r)^{-1} I_1(\kappa r) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^{2k} k! (k+1)!} (\kappa r)^{2k},$$

which according to (A.4), with $\alpha = -1$, represents a fundamental solution. $\square$

Let us finally consider the role of the terms in the kernel of the parametrix which have been excluded from our construction, i.e.,

$$K^{(1)}(r, \phi | \tilde{r}, \tilde{\phi}) = \sum_0^{\infty} r^p K_p^{(1)}(r, \phi | \tilde{r}, \tilde{\phi}).$$

Remark 5. According to our discussion after Lemma 2, it is only the $\ell = 0$ term for which the Limit $\tilde{r} \rightarrow 0$ cannot be performed.

For n even, $k_{2m}^{(1)} := P_0 K_{2m}^{(1)}$, $m = 1, 2, \dots$, is given by

$$\begin{aligned} k_{2m}^{(1)}(r, \tilde{r}) &:= -r^{2-n} \sum_{j=1}^{q_0} \delta(w_j^{(2)}(\lambda_0) - n) \ln(\tilde{r}) \left(\text{Res}_{w_j^{(2)}(\lambda_0)}^{(1)} h_{2m}^{(-1)}(\cdot, \lambda_0) \right) p_0 \\ &= -r^{2-n} \ln(\tilde{r}) \kappa^{2m} Q_m(n) p_0 \end{aligned}$$

which sums up to

$$k^{(1)}(r, \tilde{r}) := \sum_{m=1}^{\infty} r^{2m} k_{2m}^{(1)}(r, \tilde{r}) = - \left(\sum_{m=1}^{\infty} r^{2(m+1)-n} \kappa^{2m} Q_m(n) \right) \ln(\tilde{r}) p_0,$$

in particular for $n = 4$, we get

$$\begin{aligned} k^{(1)}(r, \tilde{r}) &= \left(\sum_{m=1}^{\infty} r^{2m-2} \kappa^{2m} \frac{1}{2^{2m} m! (m-1)!} \right) \ln(\tilde{r}) p_0 \\ &= \frac{\kappa^2}{4} \left(\sum_{k=0}^{\infty} (\kappa r)^{2k} \frac{1}{2^{2k} (k+1)! k!} \right) \ln(\tilde{r}) p_0 \\ &= \frac{1}{2} r^{-1} I_1(\kappa r) \ln(\tilde{r}) p_0 \end{aligned}$$

In the second line, we used (A.3), which shows that $k^{(1)}$ corresponds to the kernel of a Green operator which maps onto a smooth function in the kernel of the shifted Laplace operator.

3.2 Green's functions from the parametrix

In order to demonstrate that one can even get a generalized Laplace expansion of a Green's function from the kernel of the parametrix, we consider the case $n = 3$ where no logarithmic terms show up in the expansion.

Proposition 3. For $\Delta_3 - \kappa^2$, the kernel of the asymptotic parametrix

$$K_\kappa(r, \phi | \tilde{r}, \tilde{\phi}) = \sum_0^\infty r^p K_p(r, \phi | \tilde{r}, \tilde{\phi}).$$

is given by

$$\begin{aligned} K_\kappa(r, \phi | \tilde{r}, \tilde{\phi}) &= H(r - \tilde{r}) \frac{\pi}{2} \sum_{\ell=0}^\infty (-1)^\ell \left[\frac{I_{\ell+\frac{1}{2}}(\kappa r)}{\sqrt{r}} \frac{I_{-\ell-\frac{1}{2}}(\kappa \tilde{r})}{\sqrt{\tilde{r}}} - \frac{I_{-\ell-\frac{1}{2}}(\kappa r)}{\sqrt{r}} \frac{I_{\ell+\frac{1}{2}}(\kappa \tilde{r})}{\sqrt{\tilde{r}}} \right] p_\ell(\phi, \tilde{\phi}) \\ &\quad - \sqrt{\frac{\pi \kappa}{2}} \sum_{\ell=0}^\infty \frac{I_{\ell+\frac{1}{2}}(\kappa r)}{\sqrt{r}} S_\ell(\kappa \tilde{r}) p_\ell(\phi, \tilde{\phi}) \end{aligned} \quad (3.5)$$

with

$$S_\ell(\kappa \tilde{r}) := \sum_{m=0}^{\lfloor \ell/2 \rfloor} (\kappa \tilde{r})^{-1-\ell+2m} \frac{(-1)^m (2(\ell-m)-1)!!}{2^m m!!}.$$

and $I_{\pm\ell\pm\frac{1}{2}}$ denote modified Bessel function of first order. The corresponding family of regular distributions

$$K_\kappa(\cdot, \tilde{x}) := \begin{cases} \text{Pf. } K_\kappa(\cdot | \tilde{r}, \tilde{\phi}) & \text{for } \tilde{x} = \varphi(\tilde{r}, \tilde{\phi}) \neq 0 \\ \text{Pf. } \lim_{\tilde{r} \rightarrow 0} K_\kappa(\cdot | \tilde{r}, \tilde{\phi}) & \text{for } \tilde{x} = 0 \end{cases},$$

satisfies Definition 2 for a Green's function.

Proof. Details of the calculation for (3.5) are given in Appendix C. It is instructive to verify (3.5) by comparison with the canonical Green's function of the shifted Laplace operator, cf. [21], which can be obtained from the fundamental solution (3.3)

$$G_\kappa(x, \tilde{x}) = -\frac{1}{4\pi|x-\tilde{x}|} e^{-\kappa|x-\tilde{x}|}$$

For this Green's function a generalized Laplace expansion based on Gegenbauers addition theorem is known in the literature, cf. [22][p.366], such that

$$G_\kappa(\cdot, \tilde{x}) := \begin{cases} \text{Pf. } G_\kappa(\cdot | \tilde{r}, \tilde{\phi}) & \text{for } \tilde{x} = \varphi(\tilde{r}, \tilde{\phi}) \neq 0 \\ \text{Pf. } \lim_{\tilde{r} \rightarrow 0} G_\kappa(\cdot | \tilde{r}, \tilde{\phi}) & \text{for } \tilde{x} = 0 \end{cases}$$

with

$$\begin{aligned} G_\kappa(r, \phi | \tilde{r}, \tilde{\phi}) &= -H(r - \tilde{r}) \sum_{\ell=0}^\infty \frac{K_{\ell+\frac{1}{2}}(\kappa r)}{\sqrt{r}} \frac{I_{\ell+\frac{1}{2}}(\kappa \tilde{r})}{\sqrt{\tilde{r}}} p_\ell(\phi, \tilde{\phi}) \\ &\quad - H(r - \tilde{r}) \sum_{\ell=0}^\infty \frac{I_{\ell+\frac{1}{2}}(\kappa r)}{\sqrt{r}} \frac{K_{\ell+\frac{1}{2}}(\kappa \tilde{r})}{\sqrt{\tilde{r}}} p_\ell(\phi, \tilde{\phi}) \end{aligned} \quad (3.6)$$

where

$$K_{\ell+\frac{1}{2}}(\kappa \tilde{r}) = \frac{\pi}{2} (-1)^\ell \left(I_{-\ell-\frac{1}{2}}(\kappa r) - I_{\ell+\frac{1}{2}}(\kappa r) \right) \quad (3.7)$$

denotes a modified Bessel function of second order.

Let us consider the differences between the kernel functions (3.5) and (3.6), which is given by

$$K_\kappa(r, \phi | \tilde{r}, \tilde{\phi}) - G_\kappa(r, \phi | \tilde{r}, \tilde{\phi}) = \sum_{\ell=0}^{\infty} \frac{I_{\ell+\frac{1}{2}}(\kappa r)}{\sqrt{r}} \left(\frac{K_{\ell+\frac{1}{2}}(\kappa \tilde{r})}{\sqrt{\tilde{r}}} + \sqrt{\frac{\pi \kappa}{2}} S_\ell(\kappa \tilde{r}) \right) p_\ell(\phi, \tilde{\phi}). \quad (3.8)$$

According to (A.3), the distribution corresponding to this difference maps $\mathcal{D}(\mathbb{R}^3)$ into a smooth function in the kernel of $\Delta_3 - \kappa^2$. $\square$

Although, the kernel of the parametrix (3.5) satisfies our definition of a Green's function it is clear, by comparison with (3.6), that it has some shortcomings. First of all, (3.5) is not a symmetric kernel function, unlike (3.6), which reflects the fact that the shifted Laplace operator is symmetric. Given (3.5), this can be cured however by a simple consideration. First of all, the second sum can be skipped, because it maps $\mathcal{D}(\mathbb{R}^3)$ into smooth functions in the kernel of $\Delta_3 - \kappa^2$. Furthermore, according to the previous remark, we are free to add a term of the form

$$\frac{\pi}{2} (-1)^\ell \sum_{\ell=0}^{\infty} \frac{I_{\ell+\frac{1}{2}}(\kappa r)}{\sqrt{r}} \frac{A(\kappa \tilde{r})}{\sqrt{\tilde{r}}} p_\ell(\phi, \tilde{\phi}),$$

in order to restore symmetry under permutation $(r, \phi) \leftrightarrow (\tilde{r}, \tilde{\phi})$. Permutational symmetry requires

$$-\frac{I_{-\ell-\frac{1}{2}}(\kappa r)}{\sqrt{r}} \frac{I_{\ell+\frac{1}{2}}(\kappa \tilde{r})}{\sqrt{\tilde{r}}} + \frac{I_{\ell+\frac{1}{2}}(\kappa r)}{\sqrt{r}} \left(\frac{I_{-\ell-\frac{1}{2}}(\kappa \tilde{r})}{\sqrt{\tilde{r}}} + \frac{A(\kappa \tilde{r})}{\sqrt{\tilde{r}}} \right) = \frac{A(\kappa r)}{\sqrt{r}} \frac{I_{\ell+\frac{1}{2}}(\kappa \tilde{r})}{\sqrt{\tilde{r}}}$$

which is obviously satisfied by

$$A(\kappa \tilde{r}) = -I_{-\ell-\frac{1}{2}}(\kappa \tilde{r}) \mod I_{\ell+\frac{1}{2}}(\kappa \tilde{r})$$

The remaining ambiguity can be resolved by another shortcomming of (3.5) which concerns its asymptotic behaviour for $r, \tilde{r} \rightarrow \infty$. Whereas (3.6) decays exponentially, this is not the case for (3.5). The reason is due to the fact that the modified Bessel functions of the first kind $I_{\pm\ell\pm\frac{1}{2}}$ increase exponentially, and it is only their difference, the modified Bessel functions of the second kind $K_{\ell+\frac{1}{2}}$, cf. (3.7), which decays exponentially [1][9.7.1, 9.7.2]. In order to get the desired asymptotic behaviour, we have to add the term

$$\frac{\pi}{2} (-1)^\ell \sum_{\ell=0}^{\infty} \frac{I_{\ell+\frac{1}{2}}(\kappa r)}{\sqrt{r}} \frac{I_{\ell+\frac{1}{2}}(\kappa \tilde{r})}{\sqrt{\tilde{r}}} p_\ell(\phi, \tilde{\phi}),$$

which obviously preserves permutational symmetry. With these modification (3.5) ultimately becomes (3.6). From the point of view of singular analysis the required modifications can be assigned to Green operators, which is linked to the fact that these terms map into spaces with fixed asymptotic behaviour.

4 Application in Physics: Scattering theory

In pseudo-differential calculus the notion of ellipticity poses severe restrictions on the partial differential operators for which a parametrix exists and therefore excludes many interesting cases. A typical example, already mentioned in Section 1.1, is the Laplace operator which does not satisfy

the exit condition. As a possible remedy, one can consider the elliptic shifted Laplace operator, calculate a fundamental solution or Green's function and finally takes the limit $\kappa \rightarrow 0$, which yields the corresponding quantities for the Laplace operator, cf. Appendix A for further details.

Such an approach can be easily generalized by performing analytic continuations with respect to a given parameter on which the fundamental solution or Green's function depends. The method of analytical continuation paves the way to many interesting applications in physics, in particular quantum many-body and scattering theory.

Remark 6. *Typical differential operators in scattering theory do not satisfy the exit condition for ellipticity. A prominent example is the Helmholtz equation*

$$(\Delta + \kappa^2)u = \delta$$

which differs from the shifted Laplace equation, discussed in Section 3, by the minus sign in front of constant κ^2 . Actually, the two equations and their fundamental solutions are related by analytic continuation of the parameter κ , i.e., a rotation in the complex plane by $\pm\pi/2$ transforms one into the other. More explicitly, let us consider the fundamental solution of the Helmholtz equation in n -dimension, cf. ([6]),

$$u_{n,\kappa} = \text{Pf.} \frac{1}{4i(2\pi)^{\frac{n-2}{2}}} k^{\frac{n-2}{2}} r^{\frac{2-n}{2}} H_{\frac{n-2}{2}}^{(1)}(\kappa r),$$

where $H_{\frac{n-2}{2}}^{(1)}$ denotes a Hankel function of the first kind. Using the identity, cf. [1][9.6.4],

$$H_{\frac{n-2}{2}}^{(1)}(\kappa r) = -\frac{2i}{\pi} e^{-\frac{1}{2}\frac{n-2}{2}\pi i} K_{\frac{n-2}{2}}(-i\kappa r),$$

the fundamental solution becomes

$$u_{n,\kappa} = \text{Pf.} -(2\pi)^{-\frac{n}{2}} (-i\kappa)^{\frac{n-2}{2}} r^{\frac{2-n}{2}} K_{\frac{n-2}{2}}(-i\kappa r)$$

which can be obtained from the standard fundamental solution of the shifted Laplacian, cf. (A.1) by rotating κ from the real to the negative imaginary axis. The renormalized fundamental solution (3.4), which we have obtained from the asymptotic parametrix construction, differs from it by holomorphic terms only. Therefor it is possible to get access to a fundamental solution of the Helmholtz equation by an analytic continuation from the asymptotic parametrix construction.

Analytic continuation, like the one discussed in the previous remark is a versatile tool in quantum theory [17]. A rigorous treatment however, requires full control on the convergence of the power series expansion of the asymptotic parametrix construction. We want to discuss this topic in the simplest non trivial case of $n = 3$ nonrelativistic electron-nucleus scattering based on a differential operator, given in spherical coordinates by

$$\tilde{\mathcal{A}} := -2(\mathcal{H} + \kappa^2) = \frac{1}{r^2} \left[\left(-r \frac{\partial}{\partial r} \right)^2 - \left(-r \frac{\partial}{\partial r} \right) + \Delta_{S^2} + r^2 Z - r^2 2\kappa^2 \right], \quad (4.1)$$

where $\mathcal{H}$ denotes the non relativistic Hamiltonian of an electron (atomic units), interacting with a nucleus of charge Z , cf. [16] for further details. The corresponding Mellin symbol h of the differential operator $\mathcal{A}$ is given by the expansion

$$h(w) = h_0(w) + rh_1 + r^2 h_2 \quad \text{with } h_0(w) = w^2 - w + \Delta_{S^2}, \quad h_1 = 2Z, \quad h_2 = -2\kappa^2.$$

According to (2.5), we get for the asymptotic parametrix symbol of order 1

$$h_1^{(-1)}(w) = -(h_0^{(-1)}(w-1))Zh_0^{(-1)}(w) \quad \text{with } h_0^{(-1)}(w) = \frac{1}{h_0(w-2)}$$

and for order $p \geq 2$, the recursive formula

$$h_p^{(-1)}(w) = -\left((h_{p-1}^{(-1)}(w-1))Z - (h_{p-2}^{(-1)}(w-2))\kappa^2\right)h_0^{(-1)}(w) \quad (4.2)$$

In order to keep track of the combinatorics, let us introduce the notion of binary words given by the following definition

Definition 4. A binary word α is composed of the numbers 1 and 2, i.e., $\alpha = (\alpha_1, \alpha_2, \alpha_3, \dots)$ with $\alpha_i \in \{1, 2\}$. For a word composed of N^α characters, let N_Z^α and N_κ^α denote the number of characters 1 and 2, respectively. Furthermore, let S_p , $p \geq 1$, denote the set of all possible words such that $p = \sum_{k=1}^{N^\alpha} \alpha_k = N_Z^\alpha + 2N_\kappa^\alpha$.

According to the previous definition, we have, e.g., $S_1 = \{(1)\}$, $S_2 = \{(1,1), (2)\}$, $S_3 = \{(1,1,1), (1,2), (2,1)\}$.

Proposition 4. Given the word operations $\lfloor_1, \lfloor_2$ ($\rfloor_1, \rfloor_2$), given by $\alpha = (\dots) \rightarrow \alpha \lfloor_1 = (\dots, 1)$ ($\alpha = (\dots) \rightarrow \rfloor_1 \alpha = (1, \dots)$) and $\alpha = (\dots) \rightarrow \alpha \lfloor_2 = (\dots, 2)$ ($\alpha = (\dots) \rightarrow \rfloor_2 \alpha = (2, \dots)$), respectively. With these operations acting on all elements of a set of binary words, we get $S_p = S_{p-1} \lfloor_1 \cup S_{p-2} \lfloor_2$ and $S_p = \rfloor_1 S_{p-1} \cup \rfloor_2 S_{p-2}$, respectively.

Proof. $S_p = S_{p-1} \lfloor_1 \cup S_{p-2} \lfloor_2$ follows from the fact that $\alpha \in S_p$ is either of the form $(\dots, 1)$ or of the form $(\dots, 2)$. The other one follows in an analogous manner. $\square$

Proposition 5. The p 'th order symbol of the asymptotic parametrix is given by

$$h_p^{(-1)}(w) = \sum_{\alpha \in S_p} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha} h_0^{(-1)}(w - p_j), \quad (4.3)$$

with $p_0 = p$ and $p_j = p - \sum_{k=1}^j \alpha_k$ for $j \geq 1$.

Proof. The proof goes by induction. Formula (4.3) is obviously true for $p = 1, 2$. Using (4.2) and

Proposition 4, we can perform the induction step

$$\begin{aligned}
h_{p+1}^{(-1)}(w) &= \sum_{\alpha \in S_p} (-1)^{N_Z^\alpha + 1} 2^{N^\alpha + 1} Z^{N_Z^\alpha + 1} (\kappa^2)^{N_\kappa^\alpha} \left(\prod_{j=0}^{N^\alpha} h_0^{(-1)}(w - p_j - 1) \right) h_0^{(-1)}(w) \\
&\quad + \sum_{\alpha \in S_{p-1}} (-1)^{N_Z^\alpha} 2^{N^\alpha + 1} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha + 1} \left(\prod_{j=0}^{N^\alpha} h_0^{(-1)}(w - p_j - 2) \right) h_0^{(-1)}(w) \\
&= \sum_{\alpha \in S_p \downarrow 1} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha} h_0^{(-1)}(w - p_j) \\
&\quad + \sum_{\alpha \in S_{p-1} \downarrow 2} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha} h_0^{(-1)}(w - p_j) \\
&= \sum_{\alpha \in S_{p+1}} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha} h_0^{(-1)}(w - p_j).
\end{aligned}$$

□

Let us first consider the physically most relevant case $n = 3$ and calculate the spherical limit k_p , given by (2.12), i.e.,

$$\begin{aligned}
k_p(r) &= -\frac{1}{4\pi r} \sum_{\alpha \in S_{p-2} \downarrow 2} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} (\text{Res}_3 \prod_{j=0}^{N^\alpha} h_0^{(-1)}(\cdot - p_j, \lambda_0)) \\
&\quad + \frac{1}{4\pi r} \sum_{\alpha \in S_{p-1} \downarrow 1} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \left[\ln(r) (\text{Res}_3^{(1)} \prod_{j=0}^{N^\alpha} h_0^{(-1)}(\cdot - p_j, \lambda_0)) \right. \\
&\quad \left. - (\text{Res}_3^{(2)} \prod_{j=0}^{N^\alpha} h_0^{(-1)}(\cdot - p_j, \lambda_0)) \right]
\end{aligned} \tag{4.4}$$

Taking into account

$$\prod_{j=0}^{N^\alpha} h_0^{(-1)}(\cdot - p_j, \lambda_0) = \prod_{j=0}^{N^\alpha} \frac{1}{(w - 2 - p_j)(w - 3 - p_j)},$$

we get the resolvents

$$\begin{aligned}
\text{Res}_3 \prod_{j=0}^{N^\alpha} h_0^{(-1)}(\cdot - p_j, \lambda_0) &= \prod_{j=0}^{N^\alpha - 1} \frac{1}{p_j(p_j - 1)} \quad (\alpha \in S_{p-2} \downarrow 2 \text{ and } p \geq 2), \\
\text{Res}_3^{(1)} \prod_{j=0}^{N^\alpha} h_0^{(-1)}(\cdot - p_j, \lambda_0) &= \begin{cases} -1 & p = 1 \\ -\prod_{j=0}^{N^\alpha - 2} \frac{1}{p_j(p_j - 1)} & p \geq 2 \end{cases} \quad (\alpha \in S_{p-1} \downarrow 1),
\end{aligned}$$

$$\text{Res}_3^{(2)} \prod_{j=0}^{N^\alpha} h_0^{(-1)}(\cdot - p_j, \lambda_0) = \begin{cases} 0 & p = 1 \\ -\prod_{j=0}^{N^\alpha-2} \frac{1}{p_j(p_j-1)} \sum_{j=0}^{N^\alpha-2} \left(\frac{1}{p_j-1} + \frac{1}{p_j} \right) & p \geq 2 \end{cases} \quad (\alpha \in S_{p-1}[1]).$$

Plugging these expressions into (4.4), we get

$$k_0(r) = -\frac{1}{4\pi r}, \quad k_1(r) = \frac{1}{4\pi r} 2Z \ln(r),$$

and for $p \geq 2$

$$\begin{aligned} k_p(r) &= -\frac{1}{4\pi r} \sum_{\alpha \in S_{p-2}[2]} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-1} \frac{1}{p_j(p_j-1)} \\ &\quad - \frac{1}{4\pi r} \sum_{\alpha \in S_{p-1}[1]} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-2} \frac{1}{p_j(p_j-1)} \left[\ln(r) - \sum_{j=0}^{N^\alpha-2} \left(\frac{1}{p_j-1} + \frac{1}{p_j} \right) \right] \end{aligned} \quad (4.5)$$

Lemma 3. *Let the function $k_{\mathcal{A}}(Z, \kappa, r)$ be given by the series*

$$k_{\mathcal{A}}(Z, \kappa, r) = \sum_{p=0}^{\infty} r^p k_p(r). \quad (4.6)$$

The series converges absolutely, i.e., $k_{\mathcal{A}}$ is well defined, for all values of $r \in \mathbb{R}_+^4$. For fixed $r > 0$ it is an entire function of the parameters $Z, \kappa \in \mathbb{C}$.

Proof. Let us first estimate the cardinality of a set $|S_q|$, $q \geq 0$, which is given by

$$|S_q| = \sum_{N^\kappa=0}^{\lfloor q/2 \rfloor} \binom{q - N^\kappa}{N^\kappa} = F(q),$$

where $F(q)$ is the q 'th Fibonacci number, which for large q have the asymptotic behaviour

$$F(q) \sim \frac{a^q}{\sqrt{5}} \quad \text{with } a = \frac{1 + \sqrt{5}}{2}.$$

Furthermore, we can estimate the products

$$\frac{1}{p!(p-1)!} \leq \prod_{j=0}^{N^\alpha-1} \frac{1}{p_j(p_j-1)}, \quad \prod_{j=0}^{N^\alpha-2} \frac{1}{p_j(p_j-1)} \leq \frac{1}{p!},$$

and finally the sums

$$\sum_{j=0}^{N^\alpha-2} \left(\frac{1}{p_j-1} + \frac{1}{p_j} \right) \leq H_p + H_{p-1} \sim \ln(p) + \ln(p-1) + 2\gamma$$

Taking $C := 2a \max\{|Z|, |\kappa|\}$, we can prove the lemma by performing a direct comparison test with respect to the convergent series

$$\frac{1}{4\sqrt{5}\pi} \sum_{p=2}^{\infty} r^{p-1} \left(\frac{C^{p-2}}{(p-2)!} + \frac{C^{p-1}}{(p-1)!} [|\ln(r)| + \ln(p) + \ln(p-1) + 2\gamma] \right).$$

□

⁴Actually, it represents a regular function $k_{\mathcal{A}}(Z, \kappa, z)$ in the complex z -plane cut along the negative real axis.

For (4.5) we can consider two limits with respect to the parameters Z, κ . For $Z = 0$, we just recover (3.2) of the shifted Laplacian, whereas for $\kappa = 0$ our calculations yield a fundamental solution of the Hamiltonian.

Proposition 6. *A fundamental solution of the differential operator (4.1) for $\kappa = 0$, is given by (4.5), taking $\kappa = 0$, i.e.,*

$$k_{\mathcal{A}}(Z, 0, r) = -\frac{1}{4\pi r} \left[1 - 2Zr \ln(r) + \sum_{p=2}^{\infty} \frac{(-1)^p 2^p Z^p}{p!(p-1)!} r^p (\ln(r) - (H_p + H_{p-1} - 1)) \right], \quad (4.7)$$

where $H_p := \sum_{k=1}^p \frac{1}{k}$ denotes the p -th harmonic number. Furthermore, $k_{\mathcal{H}}(r) = -\frac{1}{2}k_{\mathcal{A}}(r)$ is a fundamental solution of the corresponding Hamiltonian.

Proof. The absolute convergence of the series (4.7) for any $r > 0$ follows from Lemma 3.

In order to show $\mathcal{A}|_{\kappa=0} \text{Pf. } k_{\mathcal{A}} = \delta$, we have to consider for $w \in \mathcal{D}(\mathbb{R}^n)$, cf. Appendix B,

$$\begin{aligned} & \int_{\mathbb{R}^3} w(x) \mathcal{A}|_{\kappa=0} \text{Pf. } k_{\mathcal{A}}(Z, 0, r) dx \\ &= 4\pi \int_0^{\infty} \left[\left(1 + r \frac{\partial}{\partial r}\right)^2 w_0(r) - \left(1 + r \frac{\partial}{\partial r}\right) w_0(r) + 2rZw_0(r) \right] k_{\mathcal{A}}(Z, 0, r) dr \\ &= 4\pi \int_0^{\infty} \left[\left(r \frac{\partial}{\partial r}\right)^2 w_0(r) + \left(r \frac{\partial}{\partial r}\right) w_0(r) + 2rZw_0(r) \right] k_{\mathcal{A}}(Z, 0, r) dr, \end{aligned} \quad (4.8)$$

where $w_0(r) := \frac{1}{4\pi} \int_{S^2} \tilde{w}(r, \phi) \mu(\phi) d\phi$ with $\tilde{w} := \varphi^* w$ in $\tilde{\mathcal{D}}(C^\wedge)$. Straightforward calculations yield

$$\begin{aligned} & \int_0^{\infty} \left(r \frac{\partial}{\partial r} w_0(r) \right) r^{-1} dr = -w_0(0) \\ & \int_0^{\infty} \left(r \frac{\partial}{\partial r} \right) \left[r \frac{\partial}{\partial r} w_0(r) \right] r^{-1} dr = 0 \end{aligned}$$

and for $q = 0, 1, 2, \dots$

$$\begin{aligned} & \int_0^{\infty} \left(r \frac{\partial}{\partial r} w_0(r) \right) r^q dr = -(q+1) \int_0^{\infty} r^q w_0(r) dr \\ & \int_0^{\infty} \left(r \frac{\partial}{\partial r} w_0(r) \right) r^q \ln(r) dr = - \int_0^{\infty} ((q+1) \ln(r) + 1) r^q w_0(r) dr \\ & \int_0^{\infty} \left(r \frac{\partial}{\partial r} \right) \left[r \frac{\partial}{\partial r} w_0(r) \right] r^q dr = (q+1)^2 \int_0^{\infty} r^q w_0(r) dr \\ & \int_0^{\infty} \left(r \frac{\partial}{\partial r} \right) \left[r \frac{\partial}{\partial r} w_0(r) \right] r^q \ln(r) dr = (q+1) \int_0^{\infty} ((q+1) \ln(r) + 2) r^q w_0(r) dr \end{aligned}$$

Interting these expressions into (4.8) yields

$$\begin{aligned}
& \int_{\mathbb{R}^n} w(x) \mathcal{A} \text{Pf. } k_{\mathcal{A}}(Z, 0, r) dx \\
&= w_0(0) + 2^2 Z^2 \int_0^\infty r \ln(r) w_0(r) dr - \sum_{p=2}^\infty \frac{(-1)^p 2^p Z^p}{p!(p-1)!} \left[p(p-1) \int_0^\infty r^{p-1} \ln(r) w_0(r) dr \right. \\
&\quad + (2p-1) \int_0^\infty r^{p-1} w_0(r) dr + 2Z \int_0^\infty r^p \ln(r) w_0(r) dr \\
&\quad \left. - (H_p + H_{p-1} - 1) \left(p(p-1) \int_0^\infty r^{p-1} w_0(r) dr + 2Z \int_0^\infty r^p \ln(r) w_0(r) dr \right) \right] \\
&= w_0(0) + \sum_{p=3}^\infty \frac{(-1)^p 2^p Z^p}{p!(p-1)!} \left[- (H_{p-1} + H_{p-2} - 1) p(p-1) \right. \\
&\quad \left. - (2p-1) + (H_p + H_{p-1} - 1) p(p-1) \right] \int_0^\infty r^{p-1} w_0(r) dr \\
&= w_0(0) \\
&= w(0)
\end{aligned}$$

□

In order to illustrate the behaviour of the fundamental solutions (4.7) of the differential operator (4.1) for $\kappa = 0$, we have plotted some of them for selected values of Z in Fig. 1. For $Z = 0$ we get a strictly increasing fundamental solution of the Laplacian as a reference function. With increasing values of Z , we observe an increasing oscillatory behaviour around this reference function. These oscillations are caused by the $2Z/r$ term in (4.1) and resemble to the oscillatory behaviour of fundamental solutions of the Helmholtz equation. In contrast to the latter, however, the oscillations are damped with increasing distance to the origin.

Next, let us consider the general case

Proposition 7. *The regular distribution $\text{Pf. } k_{\mathcal{A}}(Z, \kappa, r)$, given by (4.5) and (4.6) is a fundamental solution of the differential operator (4.1) for $Z, \kappa \in \mathbb{C}$.*

Proof. Similar to the proof of Proposition 6, we have to consider integrals

$$\begin{aligned}
& \int_{\mathbb{R}^3} w(x) \mathcal{A} \text{Pf. } k_{\mathcal{A}}(Z, \kappa, r) dx \\
&= 4\pi \int_0^\infty \left[\left(r \frac{\partial}{\partial r} \right)^2 w_0(r) + \left(r \frac{\partial}{\partial r} \right) w_0(r) + 2rZw_0(r) - 2r^2\kappa^2 w_0(r) \right] k_{\mathcal{A}}(Z, \kappa, r) dr, \quad (4.9)
\end{aligned}$$

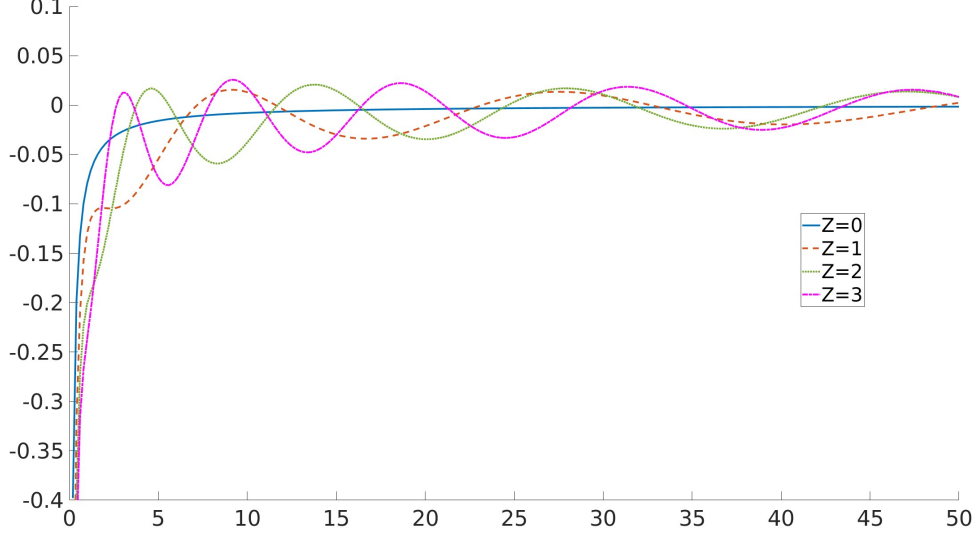


Figure 1: Fundamental solutions (4.7) of the differential operator (4.1) for $Z = 0, 1, 2, 3$ and $\kappa = 0$.

with $w \in \mathcal{D}(\mathbb{R}^3)$. Using the integrals from Proposition 6, we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^3} w(x) \mathcal{A} \text{Pf. } k_{\mathcal{A}}(Z, \kappa, r) dx \\
&= w_0(0) + 2\kappa^2 \int_0^\infty r w_0(r) dr + 2^2 Z^2 \int_0^\infty r \ln(r) w_0(r) dr - 2^2 \kappa^2 Z \int_0^\infty r^2 \ln(r) w_0(r) dr \\
&\quad - \sum_{p=2}^\infty \left\{ \sum_{\alpha \in S_{p-2} \lfloor_2} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-1} \frac{1}{p_j(p_j-1)} \left(p(p-1) \int_0^\infty r^{p-1} w_0(r) dr \right. \right. \\
&\quad \left. \left. + 2Z \int_0^\infty r^p w_0(r) dr - 2\kappa^2 \int_0^\infty r^{p+1} w_0(r) dr \right) \right. \\
&\quad \left. + \sum_{\alpha \in S_{p-1} \lfloor_1} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-2} \frac{1}{p_j(p_j-1)} \left[p(p-1) \int_0^\infty r^{p-1} \ln(r) w_0(r) dr \right. \right. \\
&\quad \left. \left. + (2p-1) \int_0^\infty r^{p-1} w_0(r) dr + 2Z \int_0^\infty r^p \ln(r) w_0(r) dr - 2\kappa^2 \int_0^\infty r^{p+1} \ln(r) w_0(r) dr \right. \right. \\
&\quad \left. \left. - \left(\sum_{j=0}^{N^\alpha-2} \left(\frac{1}{p_j-1} + \frac{1}{p_j} \right) \right) \left(p(p-1) \int_0^\infty r^{p-1} w_0(r) dr \right. \right. \right. \\
&\quad \left. \left. \left. + 2Z \int_0^\infty r^p w_0(r) dr - 2\kappa^2 \int_0^\infty r^{p+1} w_0(r) dr \right) \right] \right\}
\end{aligned}$$

In order to prove the proposition, we can compare the coefficients of the integrals $\int_0^\infty r^{p-1} w_0(r) dr$ and $\int_0^\infty r^{p-1} \ln(r) w_0(r) dr$, with $p = 1, 2, \dots$. Let us start with the integrals $\int_0^\infty r^q w_0(r) dr$. It can

be easily seen that the two $p = 2$ terms cancel each other. For $p \geq 3$ we get the coefficients

$$\begin{aligned}
& \int_0^\infty r^{p-1} w_0(r) dr : \\
& - \sum_{\alpha \in S_{p-2} \lfloor_2} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-1} \frac{1}{p_j(p_j-1)} p(p-1) \\
& - \sum_{\alpha \in S_{p-3} \lfloor_2} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-1} \frac{1}{(p-1)_j((p-1)_j-1)} 2Z \\
& + \sum_{\alpha \in S_{p-4} \lfloor_2} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-1} \frac{1}{(p-2)_j((p-2)_j-1)} 2\kappa^2 \\
& - \sum_{\alpha \in S_{p-1} \lfloor_1} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-2} \frac{1}{p_j(p_j-1)} \left[(2p-1) - \left(\sum_{j=0}^{N^\alpha-2} \left(\frac{1}{p_j-1} + \frac{1}{p_j} \right) \right) p(p-1) \right] \\
& + \sum_{\alpha \in S_{p-2} \lfloor_1} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-2} \frac{1}{(p-1)_j((p-1)_j-1)} \left(\sum_{j=0}^{N^\alpha-2} \left(\frac{1}{(p-1)_j-1} + \frac{1}{(p-1)_j} \right) \right) 2Z \\
& - \sum_{\alpha \in S_{p-3} \lfloor_1} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-2} \frac{1}{(p-2)_j((p-2)_j-1)} \left(\sum_{j=0}^{N^\alpha-2} \left(\frac{1}{(p-2)_j-1} + \frac{1}{(p-2)_j} \right) \right) 2\kappa^2
\end{aligned}$$

In order to prove, that the terms in these coefficients cancel each other, let us consider the first three terms and the last three terms, separately. The second and third term add up to

$$\begin{aligned}
& - \sum_{\alpha \in S_{p-3} \lfloor_2} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-1} \frac{1}{(p-1)_j((p-1)_j-1)} 2Z \\
& + \sum_{\alpha \in S_{p-4} \lfloor_2} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-1} \frac{1}{(p-2)_j((p-2)_j-1)} 2\kappa^2 \\
& = \sum_{\alpha \in 1 \rfloor S_{p-3} \lfloor_2} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-1} \frac{1}{p_j(p_j-1)} p(p-1) \\
& + \sum_{\alpha \in 2 \rfloor S_{p-4} \lfloor_2} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-1} \frac{1}{p_j(p_j-1)} p(p-1) \\
& = \sum_{\alpha \in S_{p-2} \lfloor_2} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-1} \frac{1}{p_j(p_j-1)} p(p-1)
\end{aligned}$$

and therefore cancel the first term. By a similar argument, we get for the last and second last term

$$\begin{aligned}
& + \sum_{\alpha \in S_{p-2} \setminus \{1\}} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-2} \frac{1}{(p-1)_j((p-1)_j-1)} \left(\sum_{j=0}^{N^\alpha-2} \left(\frac{1}{(p-1)_j-1} + \frac{1}{(p-1)_j} \right) \right) 2Z \\
& - \sum_{\alpha \in S_{p-3} \setminus \{1\}} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-2} \frac{1}{(p-2)_j((p-2)_j-1)} \left(\sum_{j=0}^{N^\alpha-2} \left(\frac{1}{(p-2)_j-1} + \frac{1}{(p-2)_j} \right) \right) 2\kappa^2 \\
& = - \sum_{\alpha \in 1 \setminus S_{p-2} \setminus \{1\}} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-2} \frac{1}{p_j(p_j-1)} \left(\sum_{j=0}^{N^\alpha-2} \left(\frac{1}{p_j-1} + \frac{1}{p_j} \right) - \frac{1}{p} - \frac{1}{p-1} \right) p(p-1) \\
& - \sum_{\alpha \in 2 \setminus S_{p-3} \setminus \{1\}} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-2} \frac{1}{p_j(p_j-1)} \left(\sum_{j=0}^{N^\alpha-2} \left(\frac{1}{p_j-1} + \frac{1}{p_j} \right) - \frac{1}{p} - \frac{1}{p-1} \right) p(p-1) \\
& = - \sum_{\alpha \in S_{p-1} \setminus \{1\}} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-2} \frac{1}{p_j(p_j-1)} \left[\left(\sum_{j=0}^{N^\alpha-2} \left(\frac{1}{p_j-1} + \frac{1}{p_j} \right) \right) p(p-1) - (2p-1) \right]
\end{aligned}$$

and therefore cancel the third last term. It remains to consider the coefficients of the integrals $\int_0^\infty r^{p-1} \ln(r) w_0(r) dr$, with $p = 2, 3, \dots$. It can be seen by inspection that the $p = 2$ and $p = 3$ terms cancel each other. For $p \geq 4$ we get the coefficients

$$\begin{aligned}
& \int_0^\infty r^{p-1} \ln(r) w_0(r) dr : \\
& - \sum_{\alpha \in S_{p-1} \setminus \{1\}} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-2} \frac{1}{p_j(p_j-1)} p(p-1) \\
& - \sum_{\alpha \in S_{p-2} \setminus \{1\}} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-2} \frac{1}{(p-1)_j((p-1)_j-1)} 2Z \\
& + \sum_{\alpha \in S_{p-3} \setminus \{1\}} (-1)^{N_Z^\alpha} 2^{N^\alpha} Z^{N_Z^\alpha} (\kappa^2)^{N_\kappa^\alpha} \prod_{j=0}^{N^\alpha-2} \frac{1}{(p-2)_j((p-2)_j-1)} 2\kappa^2
\end{aligned}$$

By the same argument given before, it can be shown that the second and third term cancel the first term, and therefore also these coefficients vanish all together. Summing up, we get

$$\int_{\mathbb{R}^n} w(x) \mathcal{A} \text{Pf. } k_{\mathcal{A}}(Z, 0, r) dx = w_0(0) = w(0),$$

which finishes the proof. $\square$

5 Conclusions

It was the intention of the present work to shed some light on possible connections between Green's functions and parametrices of elliptic partial differential operators. Closely related from a formal

point of view there are subtle differences in their conception. Roughly speaking, given a linear partial differential equation of the form $\mathcal{A}u = f$, a Green's function for $\mathcal{A}$ gives a solution via $u(x) = \int G(x, \tilde{x})f(\tilde{x})d\tilde{x}$, whereas a parametrix provides a solution u modulo some remainder, which is typically not further specified. For potential applications of parametrices in science and engineering this poses a problem and in our previous work, see e.g. [7], particular attention was paid to determine these remainders explicitly. Furthermore it is a typical feature of pseudo-differential calculus to construct parametrices in an asymptotic manner, either, within the calculus of classical pseudo-differential operators, up to a certain order in the covariables, or as in the singular case, cf. [8], up to a certain order in an appropriate distance variable. By such constructions one is left with unspecified higher order terms and the question of convergence for these asymptotic series remains open. Within the present work, we have calculated complete asymptotic series of parametrices for various variants of the Laplace operator and discussed their relation to the corresponding fundamental solutions and classical Green's functions. It turned out in these cases, that we could recover convergent series expansions for fundamental solutions and Green's functions from the parametrices. These considerations pave the way to obtain more general results which are valid for a larger class of elliptic, possibly singular differential operators. In particular we would like to overcome some severe limitations in Definition (2.1) concerning the dependence of coefficients on the variables and respective derivatives of the cone X . Eventually, we want to obtain a general framework which allows the explicit construction of Green's functions for a large class of elliptic differential operators that can be applied for numerical simulations. This seems to be a promising strategy to overcome limitations due to the presence of singularities in the underlying mathematical model.

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Conflicts of interest

The authors declare no conflicts of interest.

Appendices

A Regularization of fundamental solutions

Loosely speaking, any reasonable definition of ellipticity for differential operators in a pseudodifferential calculus implies existence of a parametrix. In the framework of the singular pseudodifferential calculus, considered in the present work, the notion of ellipticity involves a whole hierarchy of symbols associated to a differential operator, cf. Chapter 10 of [10] for a detailed discussion. For unbounded domains, e.g. $\mathbb{R}^n$, one has to take into account the exit behaviour to infinity, as a result,

the Laplacian Δ_n is not elliptic in $\mathbb{R}^n$, it is only the shifted Laplacian $\Delta_n - \kappa^2$, which satisfies the ellipticity conditions. A possible resolution to this problem is to construct at first regularized parametrices for the shifted differential operators which have the correct exit behaviour at infinity and from these, the corresponding fundamental solutions or Green's functions. Afterwards one considers the limit $\kappa \rightarrow 0$ for the regularization parameter κ , and sees whether one gets the fundamental solution or Green's function of the original problem.

Let us illustrate such an approach for the Laplacian Δ_n in $\mathbb{R}^n$ in the case $n \geq 3$. A fundamental solution of the shifted Laplacian $\Delta_n - \kappa^2$, see e.g. Schwartz [21][Section II, §3], is given by

$$u_{n,\kappa} = \text{Pf.} \left[-(2\pi)^{-\frac{n}{2}} \kappa^{\frac{n-2}{2}} r^{\frac{2-n}{2}} K_{\frac{n-2}{2}}(\kappa r) \right], \quad (\text{A.1})$$

where $K_{\frac{n-2}{2}}$, $n = 3, 4, \dots$, denote modified Bessel functions of the second kind. The modified Bessel functions have an asymptotic behaviour for $r \rightarrow 0$, cf. [1], of the form

$$K_{\frac{n-2}{2}}(\kappa r) \sim \frac{1}{2} \Gamma\left(\frac{n-2}{2}\right) \left(\frac{1}{2} \kappa r\right)^{-\frac{n-2}{2}}.$$

Using the identities

$$\Gamma\left(\frac{n-2}{2}\right) = \frac{2}{n-2} \Gamma\left(\frac{n}{2}\right) \text{ and } \omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \text{ (area of } S^{n-1}),$$

we get

$$\lim_{\kappa \rightarrow 0} u_{n,\kappa} = -\frac{1}{(n-2)\omega_n} r^{2-n}$$

which agrees with the fundamental solution of the Laplacian.

For the shifted Laplacian $\Delta_n - \kappa^2$ in even dimension n , we want to consider possible variations of the canonical fundamental solution (A.1) by subtracting some smooth terms which belong to the kernel of the shifted Laplacian. For the modified Bessel functions of integer order $K_{\frac{n-2}{2}}$, we consider the series, cf. [1][9.6.11], and obtain

$$\begin{aligned} -(2\pi)^{-\frac{n}{2}} \kappa^{\frac{n-2}{2}} r^{\frac{2-n}{2}} K_{\frac{n-2}{2}}(\kappa r) &= -\frac{1}{4\pi^{\frac{n}{2}}} r^{2-n} \sum_{k=0}^{\frac{n-4}{2}} (-1)^k \frac{(\frac{n-4}{2} - k)!}{4^k k!} (\kappa r)^{2k} \\ &\quad + (-1)^{\frac{n-2}{2}} (2\pi)^{-\frac{n}{2}} \kappa^{\frac{n-2}{2}} r^{\frac{2-n}{2}} I_{\frac{n-2}{2}}(\kappa r) \ln\left(\frac{1}{2} \kappa r\right) \\ &\quad + (-1)^{\frac{n}{2}} \kappa^{n-2} (4\pi)^{-\frac{n}{2}} \sum_{k=0}^{\infty} \left(\psi(k+1) + \psi\left(\frac{n}{2} + k\right) \right) \frac{1}{4^k k! (\frac{n-2}{2} + k)!} (\kappa r)^{2k} \end{aligned} \quad (\text{A.2})$$

with Psi function

$$\psi(1) = -\gamma, \quad \psi(m) = -\gamma + \sum_{j=1}^{m-1} \frac{1}{j}, \quad m = 2, 3, \dots$$

The function, cf. [1][9.6.10],

$$r^{\frac{2-n}{2}} I_{\frac{n-2}{2}}(\kappa r) = \left(\frac{\kappa}{2}\right)^{\frac{n-2}{2}} \sum_{k=0}^{\infty} \frac{1}{4^k k! (\frac{n-2}{2} + k)!} (\kappa r)^{2k} \quad (\text{A.3})$$

can be obviously extended to a smooth function in $\mathbb{R}^n$, which belongs to the kernel of the shifted Laplacian as can be seen by an explicit calculation in polar coordinates

$$\begin{aligned}
(\tilde{\Delta}_n - \kappa^2) r^{\frac{2-n}{2}} I_{\frac{n-2}{2}}(\kappa r) &= \frac{1}{r^2} \left[\left(-r \frac{\partial}{\partial r} \right)^2 - (n-2) \left(-r \frac{\partial}{\partial r} \right) + \Delta_{S^{n-1}} \right] r^{\frac{2-n}{2}} I_{\frac{n-2}{2}}(\kappa r) \\
&= r^{-\frac{n+1}{2}} \left[(\kappa r)^2 \partial_r^2 I_{\frac{n-2}{2}}(\kappa r) + (\kappa r) \partial_r I_{\frac{n-2}{2}}(\kappa r) - \left(\left(\frac{n-2}{2} \right)^2 + (\kappa r)^2 \right) I_{\frac{n-2}{2}}(\kappa r) \right] \\
&= 0,
\end{aligned}$$

which continuously extends to $\mathbb{R}^n$ because of the smoothness of the function. Therefore, we can subtract from (A.2) the terms

$$(-1)^{\frac{n-2}{2}} (2\pi)^{-\frac{n}{2}} \kappa^{\frac{n-2}{2}} r^{\frac{2-n}{2}} I_{\frac{n-2}{2}}(\kappa r) \ln\left(\frac{1}{2}\kappa\right)$$

and

$$-(2\gamma + \alpha)(-1)^{\frac{n}{2}} \kappa^{n-2} (4\pi)^{-\frac{n}{2}} \sum_{k=0}^{\infty} \frac{1}{4^k k! \left(\frac{n-2}{2} + k \right)!} (\kappa r)^{2k} \quad \alpha \in \mathbb{R}$$

in order to obtain the modified fundamental solution $\text{Pf}.k_{\text{mod}}(r)$ with

$$\begin{aligned}
k_{\text{mod}}(r) &= -\frac{1}{4\pi^{\frac{n}{2}}} r^{2-n} \sum_{k=0}^{\frac{n-4}{2}} (-1)^k \frac{\left(\frac{n-4}{2} - k \right)!}{4^k k!} (\kappa r)^{2k} \\
&\quad + (-1)^{\frac{n-2}{2}} (2\pi)^{-\frac{n}{2}} \kappa^{\frac{n-2}{2}} r^{\frac{2-n}{2}} I_{\frac{n-2}{2}}(\kappa r) \ln(r) \\
&\quad + (-1)^{\frac{n}{2}} \kappa^{n-2} (4\pi)^{-\frac{n}{2}} \sum_{k=0}^{\infty} \left(-\alpha + \sum_{j=1}^k j^{-1} + \sum_{j=1}^{\frac{n-2}{2}+k} j^{-1} \right) \frac{1}{4^k k! \left(\frac{n-2}{2} + k \right)!} (\kappa r)^{2k}
\end{aligned} \tag{A.4}$$

B Derivatives of distributions in polar coordinates

Within the present work, we consider regular distributions $\text{Pf}.k$ in $\mathcal{D}'(\mathbb{R}^n)$, with functions $k : C^\wedge(X) \rightarrow \mathbb{R}$ which are $\mathcal{O}(r^{2-n})$ for $r \rightarrow 0$. In order to show that $\text{Pf}.k$ represents a fundamental solution of a partial differential operator $\mathcal{A}$, with corresponding operator $\tilde{\mathcal{A}}$ in the cone algebra, we have to consider the distributional derivative $\mathcal{A}\text{Pf}.k$, i.e., the distribution defined by

$$\int_{\mathbb{R}^n} g(x) \mathcal{A}\text{Pf}.k(x) dx := \int_{\mathbb{R}^n} (\mathcal{A}^* g(x)) \text{Pf}.k(x) dx,$$

where $\mathcal{A}^*$ denotes the formally adjoint operator and g any test function in $\mathcal{D}(\mathbb{R}^n)$. It should be emphasized, that this definition refers exclusively to cartesian coordinates with corresponding Lebesgue measure dx , cf. Schwartz [21][Section II, §3].

Within the present work, we consider partial differential operators $\mathcal{A}$ which correspond to partial differential operators on the stretched cone of the form

$$\tilde{\mathcal{A}} = r^{-2} \left[\left(-r \frac{\partial}{\partial r} \right)^2 + a_1^{(0)} \left(-r \frac{\partial}{\partial r} \right) + a_0^{(0)} + b^{(0)} \Lambda_X \right],$$

with coefficients satisfying $a_0^{(0)} = -a_1^{(0)}(n-2) - (n-2)^2$. Using $(-r \frac{\partial}{\partial r}) = -(\sum_{i=1}^n x_i \frac{\partial}{\partial x_i})$ and the essential self-adjointness of Λ_X , we get

$$\begin{aligned}
& \int_{\mathbb{R}^n} g(x) \mathcal{A} \text{Pf.} k(x) dx \\
&= \int_{\mathbb{R}^n} \left(\left[\left(\sum_{i=1}^n \frac{\partial}{\partial x_i} x_i \cdot \right) \left(\sum_{i=1}^n \frac{\partial}{\partial x_j} x_j \cdot \right) - a_1^{(0)} \left(\sum_{i=1}^n \frac{\partial}{\partial x_i} x_i \cdot \right) + a_0^{(0)} + b^{(0)} \Lambda_X \right] \frac{g(x)}{|x|^2} \right) \text{Pf.} k(x) dx \\
&= \int_X \int_0^\infty \left(\left[\left(n + r \frac{\partial}{\partial r} \right)^2 - a_1^{(0)} \left(n + r \frac{\partial}{\partial r} \right) + a_0^{(0)} + b^{(0)} \Lambda_X \right] \frac{\tilde{g}(r, \phi)}{r^2} \right) k(r, \phi) r^{n-1} dr \mu(\phi) d\phi \\
&= \int_X \int_0^\infty \frac{1}{r^2} \left(\left[\left((n-2) + r \frac{\partial}{\partial r} \right)^2 + a_1^{(0)} \left((n-2) + r \frac{\partial}{\partial r} \right) \right. \right. \\
&\quad \left. \left. + a_0^{(0)} + b^{(0)} \Lambda_X \right] \tilde{g}(r, \phi) \right) k(r, \phi) r^{n-1} dr \mu(\phi) d\phi
\end{aligned}$$

with $\tilde{g} := \varphi^* g(r, \phi)$ for $g \in \mathcal{D}(\mathbb{R}^n)$, cf. Section 1. These integrals exist, because the integrand remains bounded due to our constraint on the coefficients and the $\mathcal{O}(r^{2-n})$ behaviour of k .

C Calculation of the parametrix for the shifted Laplace operator in three dimensions

In this appendix we want to give some technical details concerning the calculation of the kernel of the parametrix (3.5) for the shifted Laplace operator in three dimensions. According to Proposition 1 all poles of symbols of the parametrix are simple, and formula (2.11) simplifies to

$$\begin{aligned}
K_{2m}(r, \phi | \tilde{r}, \tilde{\phi}) &= -H(r - \tilde{r}) \sum_{\ell=0}^{\infty} \sum_{k=1}^{j_\ell} r^{2-w_k(\lambda_\ell)} \tilde{r}^{w_k(\lambda_\ell)-n} (\text{Res}_{w_k(\lambda_\ell)} h_{2m}^{(-1)}(\cdot, \lambda_\ell)) p_\ell(\phi | \tilde{\phi}) \\
&\quad + H(\tilde{r} - r) \sum_{\ell=0}^{\infty} \sum_{k=j_\ell+1}^{4m+2} r^{2-w_k(\lambda_\ell)} \tilde{r}^{w_k(\lambda_\ell)-n} (\text{Res}_{w_k(\lambda_\ell)} h_{2m}^{(-1)}(\cdot, \lambda_\ell)) p_\ell(\phi | \tilde{\phi}).
\end{aligned}$$

The integration contour $\Gamma_{\frac{7}{2}-\gamma}$ is taken with $\frac{1}{2} < \gamma < \frac{3}{2}$, where the poles

$w_{1,j}(\lambda_\ell) = 3 + \ell + p_j$, with $j = 0, 1, \dots, m$ and $w_{2,j}(\lambda_\ell) = 2 - \ell + p_j$, with $j = 0, 1, \dots, m - \lfloor \frac{\ell}{2} \rfloor - 1$

are located right of the integration contour, and the poles

$$w_{2,j}(\lambda_\ell) = 2 - \ell + p_j, \text{ with } j = m - \lfloor \frac{\ell}{2} \rfloor, \dots, m,$$

are located left of it. Here and in following, we use the notation $p_j := 2m - 2j$.

The symbols (3.1) of the parametrix are given by

$$h_{2m}^{(-1)}(w, \lambda_\ell) = \kappa^{2m} \prod_{j=0}^m \frac{1}{(w - 2 + \ell - p_j)(w - 3 - \ell - p_j)} \quad \text{for } m = 0, 1, \dots,$$

and we have to calculate the residues at their poles. Let us first consider the case $r > \tilde{r}$, where the poles right to the integration contour contribute. For these poles, we get

$$\text{Res}_{w_j(\lambda_\ell)} h_{2m}^{(-1)}(\cdot, \lambda_\ell) = \prod_{\substack{k=0 \\ k \neq j}}^m \frac{1}{p_j - p_k} \prod_{k=0}^m \frac{1}{p_j - p_k + 1 + 2\ell} \quad \text{for } j = 0, 1, \dots, m \quad w_j(\lambda_\ell) = 3 + \ell + p_j,$$

and

$$\text{Res}_{w_j(\lambda_\ell)} h_{2m}^{(-1)}(\cdot, \lambda_\ell) = \prod_{\substack{k=0 \\ k \neq j}}^m \frac{1}{p_j - p_k} \prod_{k=0}^m \frac{1}{p_j - p_k - 1 - 2\ell} \quad \text{for } j = 0, 1, \dots, m - \lfloor \frac{\ell}{2} \rfloor - 1 \quad w_j(\lambda_\ell) = 2 - \ell + p_j,$$

For the products, we get

$$\prod_{\substack{k=0 \\ k \neq j}}^m \frac{1}{p_j - p_k} = \frac{(-1)^j}{2^m j! (m-j)!},$$

$$\prod_{k=0}^m \frac{1}{p_j - p_k + 1 + 2\ell} = \begin{cases} \frac{(-1)^{\ell-j}}{(2(j-\ell)-1)!! ((2(m-j+\ell)+1)!!)} & \text{for } 2j > 2\ell + 1 \\ \frac{(2(\ell-j)-1)!!}{((2(m-j+\ell)+1)!!)} & \text{for } 2j < 2\ell + 1 \end{cases},$$

$$\prod_{k=0}^m \frac{1}{p_j - p_k - 1 - 2\ell} = \begin{cases} \frac{(-1)^{\ell+j+1}}{(2(j+\ell)+1)!! ((2(m-j-\ell)-1)!!)} & \text{for } j \leq m - \ell \\ \frac{(-1)^{m+1} ((2(j+\ell-m)-1)!!)}{(2(j+\ell)+1)!!} & \text{for } j \geq m - \ell + 1 \end{cases}.$$

We can now sum up the kernel of the parametrix,

$$\begin{aligned} & K(r, \phi | \tilde{r}, \tilde{\phi}) \\ &= - \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=0}^m r^{-1-\ell+2j} \tilde{r}^{2(m-j)+\ell} \kappa^{2m} \frac{(-1)^j f_j}{2^m j! (m-j)! ((2(m-j+\ell)+1)!!)} p_\ell(\phi | \tilde{\phi}) \\ & \quad - \sum_{\ell=0}^{\infty} \sum_{m=\lfloor \frac{\ell}{2} \rfloor + 1}^{\infty} \sum_{j=0}^{m - \lfloor \frac{\ell}{2} \rfloor - 1} r^{\ell+2j} \tilde{r}^{-1-\ell+2(m-j)} \kappa^{2m} \frac{g_{mj}}{2^m j! (m-j)! ((2(j+\ell)+1)!!)} p_\ell(\phi | \tilde{\phi}) \end{aligned} \quad (\text{C.1})$$

with

$$f_j = \begin{cases} (2(\ell-j)-1)!! & \text{for } 0 \leq j \leq \ell \\ \frac{(-1)^{j-\ell}}{(2(j-\ell)-1)!!} & \text{for } \ell+1 \leq j \end{cases},$$

and

$$g_{mj} = \begin{cases} \frac{(-1)^{\ell+1}}{(2(m-j-\ell)-1)!!} & \text{for } 0 \leq j \leq m - \ell \\ (-1)^{j+m+1} ((2(j+\ell-m)-1)!!) & \text{for } m - \ell + 1 \leq j \leq m - \lfloor \frac{\ell}{2} \rfloor - 1 \end{cases},$$

By changing the order of summation,

$$\sum_{m=0}^{\infty} \sum_{j=0}^m \rightarrow \sum_{j=0}^{\infty} \sum_{m=j}^{\infty} \xrightarrow{\tilde{m}=m-j} \sum_{j=0}^{\infty} \sum_{\tilde{m}=0}^{\infty}$$

we can achieve a separation of the variables, the first sum in (C.1) becomes

$$\begin{aligned}
& - \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\tilde{m}=0}^{\infty} r^{-1-\ell+2j} \tilde{r}^{2\tilde{m}+\ell} \kappa^{2(\tilde{m}+j)} \frac{(-1)^j f_j}{2^{\tilde{m}+j} j! \tilde{m}! ((2(\tilde{m}+\ell)+1)!!)} p_{\ell}(\phi|\tilde{\phi}) \\
& = - \sum_{\ell=0}^{\infty} \left(\sum_{j=0}^{\ell} r^{-1-\ell+2j} \kappa^{2j} \frac{(-1)^j ((2(j-\ell)-1)!!)}{2^j j!} + \sum_{j=\ell+1}^{\infty} r^{-1-\ell+2j} \kappa^{2j} \frac{(-1)^{\ell}}{2^j j! ((2(j-\ell)-1)!!)} \right) \\
& \quad \times \left(\sum_{\tilde{m}=0}^{\infty} \tilde{r}^{2\tilde{m}+\ell} \kappa^{2\tilde{m}} \frac{1}{2^{\tilde{m}} \tilde{m}! ((2(\tilde{m}+\ell)+1)!!)} \right) p_{\ell}(\phi|\tilde{\phi}) \\
& = - \frac{\pi}{2} \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{I_{-\ell-\frac{1}{2}}(\kappa r)}{\sqrt{r}} \frac{I_{\ell+\frac{1}{2}}(\kappa \tilde{r})}{\sqrt{\tilde{r}}} p_{\ell}(\phi|\tilde{\phi}), \tag{C.2}
\end{aligned}$$

in the last line, we have identified the power series with modified Bessel function of first kind $I_{\pm\ell\pm\frac{1}{2}}$, cf. [5][Eqs. 10.53.4, 10.47.8]. The second sum in (C.1) can be rewritten as

$$\begin{aligned}
& - \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\tilde{m}=\lfloor \frac{\ell}{2} \rfloor + 1}^{\infty} r^{\ell+2j} \tilde{r}^{-1-\ell+2\tilde{m}} \kappa^{2(\tilde{m}+j)} \frac{g_{\tilde{m}j}}{2^{\tilde{m}+j} j! \tilde{m}! ((2(j+\ell)+1)!!)} p_{\ell}(\phi|\tilde{\phi}) \\
& = - \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\tilde{m}=0}^{\infty} r^{\ell+2j} \tilde{r}^{-1-\ell+2\tilde{m}} \kappa^{2(\tilde{m}+j)} \frac{g_{\tilde{m}j}}{2^{\tilde{m}+j} j! \tilde{m}! ((2(j+\ell)+1)!!)} p_{\ell}(\phi|\tilde{\phi}) \\
& \quad + \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\tilde{m}=0}^{\lfloor \frac{\ell}{2} \rfloor} r^{\ell+2j} \tilde{r}^{-1-\ell+2\tilde{m}} \kappa^{2(\tilde{m}+j)} \frac{g_{\tilde{m}j}}{2^{\tilde{m}+j} j! \tilde{m}! ((2(j+\ell)+1)!!)} p_{\ell}(\phi|\tilde{\phi}) \\
& = \sum_{\ell=0}^{\infty} \left(\sum_{j=0}^{\infty} r^{\ell+2j} \kappa^{2j} \frac{1}{2^j j! ((2(j+\ell)+1)!!)} \right) \\
& \quad \times \left(\sum_{\tilde{m}=0}^{\ell-1} \tilde{r}^{-1-\ell+2\tilde{m}} \kappa^{2\tilde{m}} \frac{(-1)^{\tilde{m}} ((2(\ell-\tilde{m})-1)!!)}{2^{\tilde{m}} \tilde{m}!} + \sum_{\tilde{m}=\ell}^{\infty} \tilde{r}^{-1-\ell+2\tilde{m}} \kappa^{2\tilde{m}} \frac{(-1)^{\ell}}{2^{\tilde{m}} \tilde{m}! ((2(\tilde{m}-\ell)-1)!!)} \right) p_{\ell}(\phi|\tilde{\phi}) \\
& \quad - \sum_{\ell=0}^{\infty} \left(\sum_{j=0}^{\infty} r^{\ell+2j} \kappa^{2j} \frac{1}{2^j j! ((2(j+\ell)+1)!!)} \right) \left(\sum_{\tilde{m}=0}^{\lfloor \frac{\ell}{2} \rfloor} \tilde{r}^{-1-\ell+2\tilde{m}} \kappa^{2\tilde{m}} \frac{(-1)^{\tilde{m}} ((2(\ell-\tilde{m})-1)!!)}{2^{\tilde{m}} \tilde{m}!} \right) p_{\ell}(\phi|\tilde{\phi}) \\
& = \sum_{\ell=0}^{\infty} \left[(-1)^{\ell} \frac{\pi}{2} \frac{I_{\ell+\frac{1}{2}}(\kappa r)}{\sqrt{r}} \frac{I_{-\ell-\frac{1}{2}}(\kappa \tilde{r})}{\sqrt{\tilde{r}}} - \sqrt{\frac{\pi \kappa}{2}} \frac{I_{\ell+\frac{1}{2}}(\kappa r)}{\sqrt{r}} S_{\ell}(\kappa \tilde{r}) \right] p_{\ell}(\phi|\tilde{\phi}), \tag{C.3}
\end{aligned}$$

with

$$\tilde{g}_{\tilde{m}j} = \begin{cases} \frac{(-1)^{\ell+1}}{(2(\tilde{m}-\ell)-1)!!} & \text{for } \ell \leq \tilde{m} \\ (-1)^{\tilde{m}+1} ((2(\ell-\tilde{m})-1)!!) & \text{for } \tilde{m} \leq \ell-1 \end{cases}.$$

Like before, we identified the power series with modified Bessel function of first kind, except of the

finit sum given by

$$S_\ell(\kappa\tilde{r}) := \sum_{m=0}^{\lfloor \ell/2 \rfloor} (\kappa\tilde{r})^{-1-\ell+2m} \frac{(-1)^m (2(\ell-m)-1)!!}{2^m m!!}.$$

Finally, let us consider the case $r < \tilde{r}$, where the poles left to the integration contour contribute, we get

$$\begin{aligned} K(r, \phi | \tilde{r}, \tilde{\phi}) &= \sum_{\ell=0}^{\infty} \sum_{m=0}^{\lfloor \frac{\ell}{2} \rfloor} \sum_{j=0}^m r^{\ell+2j} \tilde{r}^{-1-\ell+2(m-j)} \kappa^{2m} \frac{g_{mj}}{2^m j! (m-j)! ((2(j+\ell)+1)!!)} p_\ell(\phi | \tilde{\phi}) \\ &\quad \sum_{\ell=0}^{\infty} \sum_{m=\lfloor \frac{\ell}{2} \rfloor + 1}^{\infty} \sum_{j=m-\lfloor \frac{\ell}{2} \rfloor}^m r^{\ell+2j} \tilde{r}^{-1-\ell+2(m-j)} \kappa^{2m} \frac{g_{mj}}{2^m j! (m-j)! ((2(j+\ell)+1)!!)} p_\ell(\phi | \tilde{\phi}) \\ &= \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=j}^{j+\lfloor \frac{\ell}{2} \rfloor} r^{\ell+2j} \tilde{r}^{-1-\ell+2(m-j)} \kappa^{2m} \frac{g_{mj}}{2^m j! (m-j)! ((2(j+\ell)+1)!!)} p_\ell(\phi | \tilde{\phi}) \\ &= \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=j}^{j+\lfloor \frac{\ell}{2} \rfloor} r^{\ell+2j} \tilde{r}^{-1-\ell+2(m-j)} \kappa^{2m} \frac{(-1)^{j+m+1} ((2(j+\ell-m)-1)!!)}{2^m j! (m-j)! ((2(j+\ell)+1)!!)} p_\ell(\phi | \tilde{\phi}) \\ &= \sum_{\ell=0}^{\infty} \sum_{j=0}^{\infty} \sum_{\tilde{m}=0}^{\lfloor \frac{\ell}{2} \rfloor} r^{\ell+2j} \tilde{r}^{-1-\ell+2\tilde{m}} \kappa^{2(\tilde{m}+j)} \frac{(-1)^{\tilde{m}+1} ((2(\ell-\tilde{m})-1)!!)}{2^{\tilde{m}+j} j! \tilde{m}! ((2(j+\ell)+1)!!)} p_\ell(\phi | \tilde{\phi}) \\ &= - \sum_{\ell=0}^{\infty} \left(\sum_{j=0}^{\infty} r^{\ell+2j} \kappa^{2j} \frac{1}{2^j j! ((2(j+\ell)+1)!!)} \right) \left(\sum_{\tilde{m}=0}^{\lfloor \frac{\ell}{2} \rfloor} \tilde{r}^{-1-\ell+2\tilde{m}} \kappa^{2\tilde{m}} \frac{(-1)^{\tilde{m}} ((2(\ell-\tilde{m})-1)!!)}{2^{\tilde{m}} \tilde{m}!} \right) p_\ell(\phi | \tilde{\phi}) \\ &= - \sum_{\ell=0}^{\infty} \sqrt{\frac{\pi \kappa}{2}} \frac{I_{\ell+\frac{1}{2}}(\kappa r)}{\sqrt{r}} S_\ell(\kappa\tilde{r}) p_\ell(\phi | \tilde{\phi}). \end{aligned} \tag{C.4}$$

Summing up the terms (C.2), (C.3) and (C.4), we get the kernel of the parametrix (3.5).

D Ellipticity conditions for partial differential operators in the cone algebra

In this appendix, we briefly outline the ellipticity conditions for partial differential operators (2.1) in the cone algebra and resulting implications for (2.7). For a detailed treatment of ellipticity conditions in the cone algebra, we refer to the monograph [20]. Let us denote the covariables for r, ϕ by ρ and ξ , respectively. The homogeneous principal symbol of $\mathcal{A}$ is given by

$$\sigma_\psi(\mathcal{A})(r, \phi, \rho, \xi) = \frac{1}{r^2} [-a_2(r)(r\rho)^2 + b(r)\sigma_\psi^2(\Lambda)(\phi, \xi)],$$

which has to satisfy the 1. ellipticity condition **(EC1)**

$$\sigma_\psi(\mathcal{A})(r, \phi, \rho, \xi) \neq 0,$$

for all $(r, \phi) \in \mathbb{R}_+ \times X$ and $(\rho, \xi) \neq 0$. Furthermore, we have to consider the reduced principal symbol

$$\tilde{\sigma}_\psi(\mathcal{A})(r, \phi, \rho, \xi) := r^2 \sigma_\psi(\mathcal{A})(r, \phi, r^{-1} \rho, \xi) = -a_2(r) \rho^2 + b(r) \sigma_\psi^2(\Lambda)(\phi, \xi),$$

which has to satisfy the 2. ellipticity condition **(EC2)**

$$\tilde{\sigma}_\psi(\mathcal{A})(r, \phi, \rho, \xi) \neq 0,$$

for all $(r, \phi) \in \overline{\mathbb{R}}_+ \times X$ and $(\rho, \xi) \neq 0$, i.e., it does not vanish up to $r = 0$. In particular, we get with respect to (2.7) the condition $a_2^{(0)} = a_2(0) \neq 0$ and $b^{(0)} = b(0) < 0$, which enter into our definition of type-A and type-B differential operators, cf. Definition 3. For the construction of an asymptotic parametrix, the 3. ellipticity condition **(EC3)** is crucial, which states that the principal conormal symbol

$$\sigma_M(\mathcal{A})(w) := \sum_{j=0}^2 a_j(0) w^j + b(0) \Lambda_X : H^s(X) \rightarrow H^{s-2}(X) \quad (\text{D.1})$$

is a family of isomorphisms for all $w \in \Gamma_{\frac{n}{2}-\gamma}$. If this is the case, $\mathcal{A}$ is said to be elliptic with respect to the weight $\gamma \in \mathbb{R}$. Finally, we want to mention the 4. ellipticity condition **(EC4)** on the exit symbols $\sigma_e(\mathcal{A})$ and $\sigma_{\psi,e}(\mathcal{A})$ which describe the behaviour of the symbol of a differential operator for $r \rightarrow \infty$. These symbols are important concerning the Fredholm property of $\mathcal{A}$ with respect to certain weighted Sobolev spaces. Since we restrict ourselves in the present work to test functions with compact support, this condition is largely ignored in the present work.

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