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KERNEL INTERPOLATION IN SOBOLEV SPACES OF HYBRID REGULARITY

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Abstract. Kernel interpolation in tensor product reproducing kernel Hilbert spaces allows for the use of sparse grids to mitigate the curse of the dimension. Typically, besides the generic constant, only a dimension dependent power of a logarithm term enters here into complexity estimates. We show that optimized sparse grids can avoid this logarithmic factor when the interpolation error is measured with respect to Sobolev spaces of hybrid regularity. Consequently, in such a situation, the complexity of kernel interpolation does not suffer from the curse of dimension.

1. Introduction

Kernel interpolation is based on the theoretical framework provided by *reproducing kernel Hilbert spaces* [1], RKHS for short. An RKHS is a specific type of Hilbert space of functions where every function's value at any given point can be *reproduced* via its inner product with the *reproducing kernel*. Mathematically, the reproducing kernel is the Riesz representer of the point evaluation. This feature amounts to a simple and efficient way to obtain kernel-based approximations that interpolate given scattered data. To this end, the representer theorem [34] is used, which states that the interpolant can be written as a finite linear combination of the kernel function evaluated at the data points. Kernel interpolation arises in machine learning and scattered data approximation, compare [8, 20, 27, 31, 32, 33]. The quality of this approximation, i.e., the estimate of the approximation error, is well established and can be found in e.g. [8, 31].

In this article, we aim at the construction of *optimized sparse grids* for the dimension robust interpolation with respect to tensor product kernels. A fundamental contribution to sparse grid kernel interpolation has recently been provided by [21, 22]. While the sparse grid construction therein relies on a multilevel approach invoking level dependent correlation lengths of the kernel function under consideration, we will use here a kernel function of *fixed correlation length* and construct for that the sparse grid interpolant. This methodology has already been exploited and analyzed in [12], including so-called superconvergence, compare [26, 28].

Error estimates for kernel interpolation have been established so far only in classical, isotropic Sobolev spaces or in Sobolev spaces of dominating mixed derivatives, [12, 21, 31]. The resulting approximation rates are in general not independent of the underlying dimension. To remove this dependency, we consider in this article now so-called Sobolev spaces of hybrid regularity. This class of function spaces and

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their associated norms are designed to simultaneously capture two different types of regularity – isotropic smoothness and mixed smoothness. They play an important role in quantum chemistry as they describe the regularity of the Coulombic wavefunction in the electronic Schrödinger equation, see [2, 24, 35] for example. As shown in [13, 14], one can derive optimized sparse grids for such spaces and indeed obtain dimension robust approximation rates under certain circumstances. Further applications of Sobolev spaces of hybrid regularity can be found in [15] for parabolic initial boundary value problems and in [18] for homogenization problems.

The analysis in [13, 14] is based on wavelet bases for the definition of the sparse grid spaces in combination with norm equivalences that hold between a whole scale of Sobolev spaces. The same technique applies also to the Fourier transform, compare [9, 10]. In contrast, in the present article, we show how these results can be transferred to kernel interpolation where one has *no* norm equivalence. Indeed, to verify dimension robust approximation rates, it suffices to have Jackson and Bernstein inequalities, which are available for kernel interpolation. Moreover, in contrast to the approach in [13, 14], our new technique allows to substantially extend the range of the scales of Sobolev spaces where the estimates are valid.

For sake of simplicity and clearness of representation, we restrict ourselves to the simple setting of tensor products of Sobolev spaces of periodic functions on the interval $\mathbb{T} = [0, 1]$ and to equidistant point distributions. Hence, we consider the d-dimensional torus $\mathbb{T}^d$ in higher dimensions. The extension to general product domains and quasi-uniform point distributions can be derived along the lines of [12]. Note that we first consider the case of a bivariate sparse grid in order to make the ideas of the proofs clear to the reader. Afterwards, we extend the results to the multivariate setting.

The outline of this article is as follows: In Section 2, we introduce the univariate Sobolev spaces under consideration and review kernel interpolation in reproducing kernel Hilbert spaces. In Section 3, we focus on the bivariate situation. We introduce Sobolev spaces of hybrid regularity and derive related Jackson and Bernstein inequalities. Then we introduce optimized sparse grid spaces and provide error estimates for the kernel interpolation in these spaces. We extend the bivariate results to arbitrary dimension in Section 4. Concluding remarks are finally stated in Section 5.

Throughout this article, in order to avoid the repeated use of generic but unspecified constants, we denote by $C \lesssim D$ that C is bounded by a multiple of D independently of parameters on which C and D may depend. Especially, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \sim D$ as $C \lesssim D$ and $C \gtrsim D$.

2. Preliminaries

2.1. **The Sobolev space $H^s(\mathbb{T})$ of periodic functions.** Let $\mathbb{T} := [0, 1]$ be the unit interval and $H^s(\mathbb{T})$ denote the periodic Sobolev space of fractional smoothness $s \geq 0$. For $s \in \mathbb{N}$, this space can be characterized by

$$H^s(\mathbb{T}) := \big\{ f \in L^2(\mathbb{T}) : \|f^{(k)}\|_{L^2(\mathbb{T})} < \infty \\ \text{and } f^{(k-1)}(0) = f^{(k-1)}(1) \text{ for all } k = 1, \ldots, s \big\},$$

equipped with the norm

$$\|f\|_{H^s(\mathbb{T})}^2 = \left(\int_{\mathbb{T}} f(x) \, \mathrm{d}x \right)^2 + \int_{\mathbb{T}} \big(f^{(s)}(x) \big)^2 \, \mathrm{d}x < \infty.$$

Indeed, the above expression defines a norm for $H^s(\mathbb{T})$ because the integrals of the derivatives $f^{(k)}$ vanishes for all $k = 1, \ldots, s-1$ due to the periodic boundary conditions.

We need the following definition:

Definition 2.1. A *reproducing kernel* for a Hilbert space $\mathcal{H}$ of functions $u\colon \Omega \to \mathbb{R}$ with inner product $(\cdot,\cdot)_{\mathcal{H}}$ is a function $\kappa : \Omega \times \Omega \to \mathbb{R}$ such that

(1) $\kappa(\cdot, y) \in \mathcal{H}$ for all $y \in \Omega$,
(2) $u(y) = \big(u, \kappa(\cdot, y)\big)_{\mathcal{H}}$ for all $u \in \mathcal{H}$ and all $y \in \Omega$.

A Hilbert space $\mathcal{H}$ with reproducing kernel $\kappa\colon \Omega \times \Omega \to \mathbb{R}$ is called *reproducing kernel Hilbert space* (RKHS).

Reproducing kernels are known to be *symmetric* and *positive semidefinite*. Thereby, a continuous kernel $\kappa : \Omega \times \Omega \to \mathbb{R}$ is called positive semidefinite if

$$\sum_{i,j=1}^{N} \alpha_i \alpha_j \kappa(x_i, x_j) \geq 0 \tag{2.1}$$

holds for all all mutually distinct points $x_1, \ldots, x_N \in \Omega$ and all $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$, for any $N \in \mathbb{N}$. The kernel is even *positive definite* if the inequality in (2.1) is strict whenever at least one α_i is different from 0.

For general real $s > \frac{1}{2}$, the reproducing kernel in $H^s(\mathbb{T})$ is given by

$$\kappa(x, y) = 1 + \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{1}{(2\pi|k|)^{2s}} \exp\big(2\pi i k(x - y)\big). \tag{2.2}$$

It is symmetric and positive definite. For $s \in \mathbb{N}$ being a natural number, this kernel simplifies to

$$\kappa(x, y) = 1 + \frac{(-1)^{s+1}}{(2s)!} B_{2s}(|x - y|),$$

where $B_{2s} : [0, 1] \to \mathbb{R}$ denotes the Bernoulli polynomial of degree $2s$, compare [3, 5].

2.2. **Kernel interpolation.** We fix the $H^p(\mathbb{T})$ with $p > \frac{1}{2}$ as reproducing kernel Hilbert space with kernel $k(\cdot, \cdot)$ given by (2.2) for $s = p$. For $j \in \mathbb{N}_0$, we define the index set

$$\Delta_j := \{0, \ldots, 2^j - 1\}$$

and the associated equidistant grid

$$X_j := \{x_{j,k} := 2^{-j}k : k \in \Delta_j\}.$$

Then, the *kernel interpolant*

$$u_j(x) = \sum_{k \in \Delta_j} u_{j,k} \kappa(x, x_{j,k}) \in H^p(\mathbb{T})$$

with respect to the grid X_j is given by solving the linear system of equations

$$\boldsymbol{K}_j \boldsymbol{u}_j = \boldsymbol{f}_j, \tag{2.3}$$

where

$$\boldsymbol{K}_j = [\kappa(x_{j,k}, x_{j,k'})]_{k,k' \in \Delta_j}, \quad \boldsymbol{u}_j = [u_{j,k}]_{k \in \Delta_j}, \quad \boldsymbol{f}_j = [u(x_{j,k})]_{k \in \Delta_j}. \tag{2.4}$$

The matrix $\boldsymbol{K}_j$ is called *kernel matrix*. It is a periodic Toeplitz matrix, so that the linear system (2.4) of equations can efficiently be solved by the fast Fourier transform.

The kernel interpolant is known to be the *best approximation* of a given function $u \in H^p(\mathbb{T})$ in

$$V_j = \operatorname{span}\{\kappa(\cdot, x) : x \in X_j\} \subset H^p(\mathbb{T})$$

with respect to the $H^p(\mathbb{T})$-norm. This means that it solves the following Galerkin problem:

$$\text{Seek } P_j u \in V_j: \quad (P_j u, v)_{H^p(\mathbb{T})} = (u, v)_{H^p(\mathbb{T})} \quad \forall\, v \in V_j. \tag{2.5}$$

In other words, the (Galerkin) projection $P_j : H^p(\mathbb{T}) \to V_j$ is $H^p(\mathbb{T})$-orthogonal. Recall that $P_j u$ is obtained by simply solving the associated kernel system (2.3) with (2.4).

2.3. **Jackson and Bernstein estimates.** The approximation $P_j u$ satisfies the error estimate

$$\|u - P_j u\|_{L^2(\mathbb{T})} \lesssim 2^{-pj} \|u\|_{H^p(\mathbb{T})} \tag{2.6}$$

uniformly in $j \in \mathbb{N}_0$, see [31, Proposition 11.30] for example. Since there holds

$$(u, v)_{H^p(\mathbb{T})} \lesssim \|u\|_{L^2(\mathbb{T})} \|v\|_{H^{2p}(\mathbb{T})},$$

this rate of convergence can be doubled by using [28, Theorem 1] provided that the data satisfy even $u \in H^{2p}(\mathbb{T})$, i.e., we then have

$$\|u - P_j u\|_{L^2(\mathbb{T})} \lesssim 2^{-2jp} \|u\|_{H^{2p}(\mathbb{T})}. \tag{2.7}$$

In view of this result and employing Galerkin orthogonality, we find

$$\begin{aligned} \|u - P_j u\|_{H^p(\mathbb{T})}^2 &= (u - P_j u, u)_{H^p(\mathbb{T})} \\ &\lesssim \|u - P_j u\|_{L^2(\mathbb{T})} \|u\|_{H^{2p}(\mathbb{T})} \\ &\lesssim 2^{-2jp} \|u\|_{H^{2p}(\mathbb{T})}^2, \end{aligned}$$

which implies

$$\|u - P_j u\|_{H^p(\mathbb{T})} \lesssim 2^{-jp} \|u\|_{H^{2p}(\mathbb{T})}. \tag{2.8}$$

Next we consider the projection $Q_j : H^p(\mathbb{T}) \to V_j$ given by

$$Q_j u := P_j u - P_{j-1} u, \quad \text{where } P_{-1} u := 0. \tag{2.9}$$

We find from (2.7) and (2.8) the estimates

$$\|Q_j u\|_{L^2(\mathbb{T})} \lesssim 2^{-jp} \|u\|_{H^p(\mathbb{T})}, \quad \|Q_j u\|_{H^p(\mathbb{T})} \lesssim 2^{-jp} \|u\|_{H^{2p}(\mathbb{T})}.$$

By interpolation, we thus arrive at the approximation property, also known as *Jackson's inequality*,

$$\|Q_j u\|_{H^{t_1}(\mathbb{T})} \lesssim 2^{-j(t_2 - t_1)} \|u\|_{H^{t_2}(\mathbb{T})} \quad \text{for all } 0 \le t_1 \le p \le t_2 \le 2p. \tag{2.10}$$

Finally, from [25, 30], we obtain for $u \in H^s(\mathbb{T})$ the inverse inequality, also known as *Bernstein's inequality*,

$$\|P_j u\|_{H^{t_2}(\mathbb{T})} \lesssim 2^{j(t_2 - t_1)} \|P_j u\|_{H^{t_1}(\mathbb{T})} \quad \text{for all } 0 \le t_1 \le t_2 \le p,$$

which we will need in the following, immediately resulting from

$$\|Q_j u\|_{H^{t_2}(\mathbb{T})} \lesssim 2^{j(t_2 - t_1)} \|Q_j u\|_{H^{t_1}(\mathbb{T})} \quad \text{for all } 0 \le t_1 \le t_2 \le p. \tag{2.11}$$

Note the different ranges where the Jackson and Bernstein inequalities hold. Estimate (2.10) is valid whenever the function u to be approximated provides extra regularity relative to the underlying RKHS $H^p(\mathbb{T})$ and the error is measured in a weaker norm than the $H^p(\mathbb{T})$-norm. For functions $u \in H^p(\mathbb{T})$, the inverse estimate (2.11) however bounds a stronger norm of the expression $Q_j u \in V_j$ by a weaker norm of this expression, but both norms are now weaker than the $H^p(\mathbb{T})$-norm.

3. Bivariate approximation

3.1. **Sobolev spaces of hybrid regularity.** Sobolev spaces of hybrid regularity have been introduced firstly in [13]. They are given by

$$H^{s,t}_{\text{iso-mix}}(\mathbb{T}^2) := H^t(\mathbb{T}) \otimes H^{s+t}(\mathbb{T}) \cap H^{s+t}(\mathbb{T}) \otimes H^t(\mathbb{T}) \tag{3.1}$$

This class of spaces $H^{s,t}_{\text{iso-mix}}(\mathbb{T}^2)$ can be characterized for $t \in \mathbb{N}_0$ by

$$H^{s,t}_{\text{iso-mix}}(\mathbb{T}^2) := \big\{ f \in L^2(\mathbb{T}^2) : \|\partial^{\boldsymbol{\alpha}} f\|_{H^t_{\text{mix}}(\mathbb{T}^2)} < \infty \text{ for all } \|\boldsymbol{\alpha}\|_1 \leq s \big\}.$$

Thus, the functions that are contained in $H^{s,t}_{\text{iso-mix}}(\mathbb{T}^2)$ are all functions from the classical, isotropic Sobolev space $H^s_{\text{iso}}(\mathbb{T}^2)$, where the s-th order derivatives provide in addition mixed Sobolev smoothness of order t. Note that the classical, isotropic Sobolev space $H^s_{\text{iso}}(\mathbb{T}^2)$ satisfies

$$H^s_{\text{iso}}(\mathbb{T}^2) = H^{s,0}_{\text{iso-mix}}(\mathbb{T}^2) = H^s(\mathbb{T}) \otimes L^2(\mathbb{T}) \cap L^2(\mathbb{T}) \otimes H^s(\mathbb{T})$$

while the classical Sobolev space $H^t_{\text{mix}}(\mathbb{T}^2)$ of dominating mixed regularity, also called *mixed* Sobolev space, satisfies

$$H^t_{\text{mix}}(\mathbb{T}^2) = H^{0,t}_{\text{iso-mix}}(\mathbb{T}^2).$$

We refer to Figure 1 for an illustration of the different Sobolev spaces under consideration. It especially shows the obvious embeddings

$$H^{s+t}_{\text{mix}}(\mathbb{T}^2) \subset H^{s,t}_{\text{iso-mix}}(\mathbb{T}^2) \subset H^s_{\text{iso}}(\mathbb{T}^2).$$

3.2. **Bernstein and Jackson inequalities.** If $\kappa(\cdot,\cdot)$ is the reproducing kernel in $H^p(\mathbb{T})$, then the product kernel

$$\boldsymbol{\kappa}(\boldsymbol{x}, \boldsymbol{y}) := \kappa(x_1, y_1) \otimes \kappa(x_2, y_2). \tag{3.2}$$

is the reproducing kernel in the mixed Sobolev space $H^p_{\text{mix}}(\mathbb{T}^2)$. This space will serve as the reproducing kernel Hilbert space under consideration, where we carry out all our estimates in the following.

Remark 3.1. Let $\kappa_s(x, y)$ be the kernel for the Sobolev space $H^s(\mathbb{T})$ and $\kappa_t(x, y)$ be the kernel for the Sobolev space $H^t(\mathbb{T})$, where we assume that $d/2 < t \leq s$. Then, the kernel

$$\boldsymbol{\kappa}(\boldsymbol{x}, \boldsymbol{y}) := \kappa_s(x_1, y_1) \otimes \kappa_t(x_2, y_2) + \kappa_t(x_1, y_1) \otimes \kappa_s(x_2, y_2) \tag{3.3}$$

is the kernel of the Sobolev space of hybrid smoothness $H^{s-t,t}_{\text{iso-mix}}(\mathbb{T}^2)$. Note, however, that for this kernel no estimates on the approximation errors are known so far.

Given a function $u \in H^p_{\text{mix}}(\mathbb{T}^2)$, the computation of the respective kernel interpolant $u \mapsto \boldsymbol{P_j} u \in \boldsymbol{V_j} := V_{j_1} \otimes V_{j_2}$, where

$$\boldsymbol{P_j} := P_{j_1} \otimes P_{j_2} : H^p_{\text{mix}}(\mathbb{T}^2) \to \boldsymbol{V_j}$$

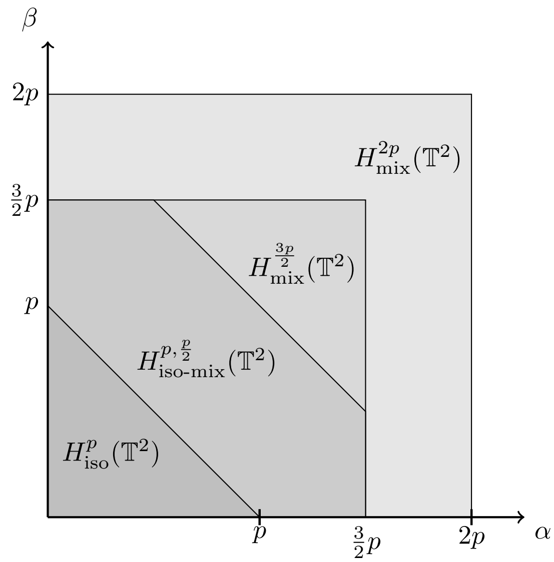

FIGURE 1. Visualization of the derivatives $\partial_x^\alpha \partial_y^\beta f(x,y)$ being bounded in the Sobolev spaces $H^{s,t}_{\text{iso-mix}}(\mathbb{T}^2)$ of hybrid regularity in comparison to the isotropic Sobolev spaces $H^{s}_{\mathrm{iso}}(\mathbb{T}^2)$ and the Sobolev spaces of dominating mixed derivatives $H^{t}_{\mathrm{mix}}(\mathbb{T}^2)$ for the specific choices of s and t.

with P_j given by (2.5), amounts to solving the linear system of equations

$$(\boldsymbol{K}_{j_1} \otimes \boldsymbol{K}_{j_2})\boldsymbol{u}_{\boldsymbol{j}} = \boldsymbol{f}_{\boldsymbol{j}}. \tag{3.4}$$

Here $\boldsymbol{K}_{j_1}$ and $\boldsymbol{K}_{j_2}$ are the univariate kernel matrices defined in (2.4), while

$$\boldsymbol{u}_{\boldsymbol{j}} = [u_{\boldsymbol{j},\boldsymbol{k}}]_{\boldsymbol{k}\in\boldsymbol{\Delta}_{\boldsymbol{j}}}, \quad \boldsymbol{f}_{\boldsymbol{j}} = [u(\boldsymbol{x}_{\boldsymbol{j},\boldsymbol{k}})]_{\boldsymbol{k}\in\boldsymbol{\Delta}_{\boldsymbol{j}}},$$

where

$$\boldsymbol{\Delta}_{\boldsymbol{j}} := \Delta_{j_1} \times \Delta_{j_2}, \quad \boldsymbol{x}_{\boldsymbol{j},\boldsymbol{k}} := (x_{j_1,k_1}, x_{j_2,k_2}).$$

Since the system matrix is a Kronecker product of periodic Toeplitz matrices, the linear system (3.4) of equations can efficiently be solved by means of the fast Fourier transform using well-known tensor and matrizication techniques found in e.g. [17].

Setting

$$\boldsymbol{Q}_{\boldsymbol{j}} := Q_{j_1} \otimes Q_{j_2} = (P_{j_1} - P_{j_1-1}) \otimes (P_{j_2} - P_{j_2-1}) : H^p_{\mathrm{mix}}(\mathbb{T}^2) \to V_{j_1} \otimes V_{j_2}$$

with Q_j given by (2.9), and using standard tensor product arguments, we obtain in view of (2.10) the approximation property

$$\|\boldsymbol{Q}_{\boldsymbol{j}} u\|_{H^{t_1}_{\mathrm{mix}}(\mathbb{T}^2)} \lesssim 2^{-(t_2-t_1)\|\boldsymbol{j}\|_1} \|u\|_{H^{t_2}_{\mathrm{mix}}(\mathbb{T}^2)} \quad \text{for all } 0 \le t_1 \le p \le t_2 \le 2p \tag{3.5}$$

and in view of (2.11) the inverse inequality

$$\|\boldsymbol{Q}_{\boldsymbol{j}} u\|_{H^{t_2}_{\mathrm{mix}}(\mathbb{T}^2)} \lesssim 2^{(t_2-t_1)\|\boldsymbol{j}\|_1} \|\boldsymbol{Q}_{\boldsymbol{j}} u\|_{H^{t_1}_{\mathrm{mix}}(\mathbb{T}^2)} \quad \text{for all } 0 \le t_1 \le t_2 \le p. \tag{3.6}$$

Lemma 3.2 (Isotropic inverse estimate)**.** *For $0 \le s_1 \le s_2 \le p$ and $u \in H^p_{\text{mix}}(\mathbb{T}^2)$, we find*

$$\|\boldsymbol{Q_j} u\|_{H^{s_2}_{\text{iso}}(\mathbb{T}^2)} \lesssim 2^{(s_2-s_1)\|\mathbf{j}\|_\infty} \|\boldsymbol{Q_j} u\|_{H^{s_1}_{\text{iso}}(\mathbb{T}^2)}$$

for all $\boldsymbol{j} \in \mathbb{N}_0^2$.

Proof. According to the univariate inverse estimate (2.11), we find

$$\begin{aligned}\|\boldsymbol{Q_j} u\|_{H^{s_2}_{\text{iso}}(\mathbb{T}^2)} &\lesssim \|\boldsymbol{Q_j} u\|_{H^{s_2}(\mathbb{T})\otimes L^2(\mathbb{T})} + \|\boldsymbol{Q_j} u\|_{L^2(\mathbb{T})\otimes H^{s_2}(\mathbb{T})}\\ &\lesssim 2^{(s_2-s_1)j_1}\|\boldsymbol{Q_j} u\|_{H^{s_1}(\mathbb{T})\otimes L^2(\mathbb{T})} + 2^{(s_2-s_1)j_2}\|\boldsymbol{Q_j} u\|_{L^2(\mathbb{T})\otimes H^{s_1}(\mathbb{T})}\\ &\lesssim 2^{(s_2-s_1)\|\boldsymbol{j}\|_\infty}\|\boldsymbol{Q_j} u\|_{H^{s_1}_{\text{iso}}(\mathbb{T}^2)}.\end{aligned}$$

□

We note that the proof applies also if we add some mixed Sobolev regularity $t \ge 0$, i.e., we have

$$\|\boldsymbol{Q_j} u\|_{H^{s_2,t}_{\text{iso-mix}}(\mathbb{T}^2)} \lesssim 2^{(s_2-s_1)\|\mathbf{j}\|_\infty} \|\boldsymbol{Q_j} u\|_{H^{s_1,t}_{\text{iso-mix}}(\mathbb{T}^2)} \tag{3.7}$$

for all $\boldsymbol{j} \in \mathbb{N}_0^2$ provided that $0 \le s_1 \le s_2 \le p - t$. Indeed, even more general, we will proof the following result.

Lemma 3.3 (General inverse estimate)**.** *For $0 \le s_1 \le s_2 \le p$ and $0 \le t_1 \le t_2 \le p$ such that $s_2 + t_2 \le p$ and $u \in H^p_{\text{mix}}(\mathbb{T}^2)$, there holds*

$$\|\boldsymbol{Q_j} u\|_{H^{s_2,t_2}_{\text{iso-mix}}(\mathbb{T}^2)} \lesssim 2^{(s_2-s_1)\|\mathbf{j}\|_\infty + (t_2-t_1)\|\mathbf{j}\|_1} \|\boldsymbol{Q_j} u\|_{H^{s_1,t_1}_{\text{iso-mix}}(\mathbb{T}^2)}$$

for all $\boldsymbol{j} \in \mathbb{N}_0^2$.

Proof. Applying (3.7) yields

$$\|\boldsymbol{Q_j} u\|_{H^{s_2,t_2}_{\text{iso-mix}}(\mathbb{T}^2)} \lesssim 2^{(s_2-s_1)\|\boldsymbol{j}\|_\infty} \|\boldsymbol{Q_j} u\|_{H^{s_1,t_2}_{\text{iso-mix}}(\mathbb{T}^2)}.$$

In view of the inverse estimate (3.6), we further have

$$\|\boldsymbol{Q_j} u\|_{H^{s_1,t_2}_{\text{iso-mix}}(\mathbb{T}^2)} \lesssim 2^{(t_2-t_1)\|\boldsymbol{j}\|_1} \|\boldsymbol{Q_j} u\|_{H^{s_1,t_1}_{\text{iso-mix}}(\mathbb{T}^2)}.$$

Putting both estimates together, we obtain the desired result. □

3.3. **Optimized sparse grid spaces.** We now discuss the construction of optimal sparse grid spaces for the approximation in the Sobolev space $H^{s,t}_{\text{iso-mix}}(\mathbb{T}^2)$ of hybrid smoothness as given in (3.1). To this end, define the *optimized sparse grid space*

$$\widehat{\boldsymbol{V}}_J^\lambda := \sum_{\boldsymbol{\lambda} \in \mathcal{I}_J^\lambda} V_{j_1} \otimes V_{j_2}, \tag{3.8}$$

where the underlying index set for the levels is given by

$$\mathcal{I}_J^\lambda = \{\boldsymbol{j} \in \mathbb{N}_0^2 : \|\boldsymbol{j}\|_1 - \lambda\|\boldsymbol{j}\|_\infty \le J(1-\lambda)\} \tag{3.9}$$

with $\lambda \in (-\infty, 1)$. Note that the choice $\lambda \to -\infty$ yields the classical full tensor product space while the choice $\lambda = 0$ results in the classical sparse grid space, compare Figure 2 for an illustration.

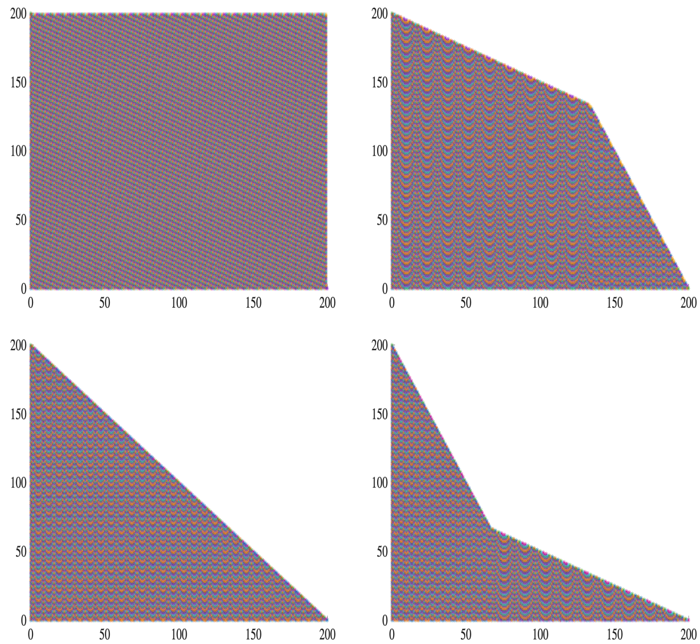

FIGURE 2. The index sets $\mathcal{I}_J^\lambda$ for $J = 200$ and $\lambda \to -\infty$ (top left), $\lambda = -1$ (top right), $\lambda = 0$ (bottom left), and $\lambda = 0.5$ (bottom right).

Theorem 3.4 (Complexity)**.** *Consider the sparse grid space* $\widehat{\boldsymbol{V}}_J^\lambda$ *given by* (3.8) *with* (3.9) *and* $\lambda \in (-\infty, 1)$. *Then there holds*

$$\dim\big(\widehat{\boldsymbol{V}}_J^\lambda\big) \lesssim \begin{cases} 2^J, & \text{if } \lambda > 0, \\ 2^J J, & \text{if } \lambda = 0, \\ 2^{2J\frac{1-\lambda}{2-\lambda}}, & \text{if } \lambda < 0. \end{cases}$$

Proof. We assume without loss of generality that $\lambda \neq 0$ as the result for the standard sparse grid is well-known, see [4] for example.

We first note that the inequality in (3.9) can be rewritten by means of $\|\boldsymbol{j}\|_1 = \min\{j_1, j_2\} + \max\{j_1, j_2\}$ and $\|\boldsymbol{j}\|_\infty = \max\{j_1, j_2\}$ as

$$\min\{j_1, j_2\} + \max\{j_1, j_2\}(1-\lambda) \leq J(1-\lambda).$$

Hence, the indices contained in $\mathcal{I}_J^\lambda$ are characterized by the inequality

$$\frac{\min\{j_1, j_2\}}{1-\lambda} + \max\{j_1, j_2\} \leq J.$$

The minimum and maximum switches at the level j that satisfies

$$\frac{j}{1-\lambda} + j = J \Rightarrow j = J\frac{1-\lambda}{2-\lambda}. \tag{3.10}$$

Hence, we can estimate[1]

$$\begin{aligned}\dim\big(\widehat{\boldsymbol{V}}_J^\lambda\big) &\lesssim \sum_{j_2=0}^{J\frac{1-\lambda}{2-\lambda}} \sum_{j_1=0}^{J-\frac{j_2}{1-\lambda}} 2^{j_1+j_2} + \sum_{j_1=0}^{J\frac{1-\lambda}{2-\lambda}} \sum_{j_2=0}^{J-\frac{j_1}{1-\lambda}} 2^{j_1+j_2}\\ &\lesssim \sum_{j_2=0}^{J\frac{1-\lambda}{2-\lambda}} 2^{j_2+J-\frac{j_2}{1-\lambda}} + \sum_{j_1=0}^{J\frac{1-\lambda}{2-\lambda}} 2^{j_1+J-\frac{j_1}{1-\lambda}}\\ &\lesssim 2^J\left[\sum_{j_2=0}^{J\frac{1-\lambda}{2-\lambda}} 2^{-j_2\frac{\lambda}{1-\lambda}} + \sum_{j_1=0}^{J\frac{1-\lambda}{2-\lambda}} 2^{-j_1\frac{\lambda}{1-\lambda}}\right].\end{aligned}$$

If $\lambda > 0$, we have always negative exponents in the sums and arrive therefore at the desired claim $\dim\big(\widehat{\boldsymbol{V}}_J^\lambda\big) \lesssim 2^J$. If $\lambda < 0$, the exponents in the sums are always positive which amounts to

$$\dim\big(\widehat{\boldsymbol{V}}_J^\lambda\big) \lesssim 2^J 2^{-J\frac{\lambda}{2-\lambda}} = 2^{2J\frac{1-\lambda}{2-\lambda}}.$$

□

We see that the logarithm in the dimension of the optimized sparse grid space $\widehat{\boldsymbol{V}}_J^\lambda$ disappears for $\lambda > 0$, whereas the dimension tends towards the dimension of the full tensor product space $V_J \otimes V_J$ for $\lambda \to -\infty$

3.4. **Approximation rates.** We shall next investigate the approximation power of the sparse grid space $\widehat{\boldsymbol{V}}_J^\lambda$. To this end, we define the projection $\widehat{\boldsymbol{Q}}_J^\lambda : H^p_{\text{mix}}(\mathbb{T}^2) \to \widehat{\boldsymbol{V}}_J^\lambda$ onto the optimized sparse grid space $\widehat{\boldsymbol{V}}_J^\lambda$ by

$$\widehat{\boldsymbol{Q}}_J^\lambda = \sum_{\boldsymbol{j}\in\mathcal{I}_J^\lambda} \boldsymbol{Q}_{\boldsymbol{j}}.$$

Theorem 3.5 (Iso-mix-convergence)**.** *Assume* $0 \le s \le p$ *and* $p \le t \le 2p$. *Then, there holds*

$$\big\|u - \widehat{\boldsymbol{Q}}_J^\lambda u\big\|_{H^s_{\text{iso}}(\mathbb{T}^2)} \lesssim 2^{-J(t-s)} \|u\|_{H^t_{\text{mix}}(\mathbb{T}^2)}$$

provided that $\lambda \in [0, \frac{s}{t})$. *If* $\lambda = \frac{s}{t}$, *then an additional logarithmic factor appears, i.e.*

$$\big\|u - \widehat{\boldsymbol{Q}}_J^\lambda u\big\|_{H^s_{\text{iso}}(\mathbb{T}^2)} \lesssim J 2^{-J(t-s)} \|u\|_{H^t_{\text{mix}}(\mathbb{T}^2)}.$$

Proof. By combining the isotropic inverse estimate from Lemma 3.2 with the approximation property (3.5) we find that

$$\begin{aligned}\|u - \widehat{\boldsymbol{Q}}_J^\lambda u\|_{H^s_{\text{iso}}(\mathbb{T}^2)} &\le \sum_{\boldsymbol{j}\notin\mathcal{I}_J^\lambda} \|\mathbf{Q}_{\boldsymbol{j}} u\|_{H^s_{\text{iso}}(\mathbb{T}^2)}\\ &\lesssim \sum_{\boldsymbol{j}\notin\mathcal{I}_J^\lambda} 2^{s\|\boldsymbol{j}\|_\infty} \|\mathbf{Q}_{\boldsymbol{j}} u\|_{L^2(\mathbb{T}^2)}\\ &\lesssim \sum_{\boldsymbol{j}\notin\mathcal{I}_J^\lambda} 2^{s\|\boldsymbol{j}\|_\infty - t\|\boldsymbol{j}\|_1} \|u\|_{H^t_{\text{mix}}(\mathbb{T}^2)}.\end{aligned}$$

[1]Note that we count here for simplicity certain indices twice. However, this does only enter into the generic constant.

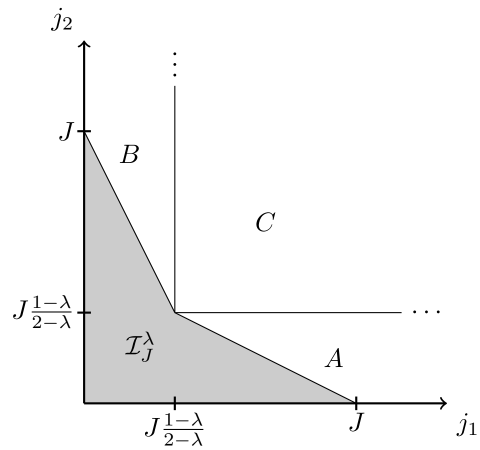

FIGURE 3. Visualization of the panelization of $\mathbb{N}_0^2$ into the index set $\mathcal{I}_J^\lambda$ and the index sets which enter the sums A, B, and C.

We obtain[2]

$$\begin{aligned}\big\|u-\widehat{\boldsymbol{Q}}_J^\lambda u\big\|_{H^s_{\mathrm{iso}}(\mathbb{T}^2)} &\lesssim \sum_{j_2=0}^{J\frac{1-\lambda}{2-\lambda}} \sum_{j_1=J-\frac{j_2}{1-\lambda}}^{\infty} 2^{s\|\boldsymbol{j}\|_\infty - t\|\boldsymbol{j}\|_1}\|u\|_{H^t_{\mathrm{mix}}(\mathbb{T}^2)} \\ &\quad+ \sum_{j_1=0}^{J\frac{1-\lambda}{2-\lambda}} \sum_{j_2=J-\frac{j_1}{1-\lambda}}^{\infty} 2^{s\|\boldsymbol{j}\|_\infty - t\|\boldsymbol{j}\|_1}\|u\|_{H^t_{\mathrm{mix}}(\mathbb{T}^2)} \\ &\quad+ \sum_{j_1=J\frac{1-\lambda}{2-\lambda}}^{\infty} \sum_{j_2=J\frac{1-\lambda}{2-\lambda}}^{\infty} 2^{s\|\boldsymbol{j}\|_\infty - t\|\boldsymbol{j}\|_1}\|u\|_{H^t_{\mathrm{mix}}(\mathbb{T}^2)} = A+B+C,\end{aligned}$$

where obviously $A=B$ holds, compare Figure 3 for an illustration.

To estimate B we use

$$\begin{aligned}B &\lesssim \sum_{j_1=0}^{J\frac{1-\lambda}{2-\lambda}} \sum_{j_2=J-\frac{j_1}{1-\lambda}}^{\infty} 2^{(s-t)j_2-tj_1}\|u\|_{H^t_{\mathrm{mix}}(\mathbb{T}^2)} \\ &\lesssim \sum_{j_1=0}^{J\frac{1-\lambda}{2-\lambda}} 2^{(s-t)(J-\frac{j_1}{1-\lambda})-tj_1} \underbrace{\sum_{j_2=0}^{\infty} 2^{(s-t)j_2}}_{\lesssim 1} \|u\|_{H^t_{\mathrm{mix}}(\mathbb{T}^2)} \\ &\lesssim 2^{(s-t)J} \sum_{j_1=0}^{J\frac{1-\lambda}{2-\lambda}} 2^{(t-s)\frac{j_1}{1-\lambda}-tj_1}\|u\|_{H^t_{\mathrm{mix}}(\mathbb{T}^2)}.\end{aligned}$$

[2]Here and in the following, the summation limits are in general no natural numbers and must of course be rounded properly. We leave this to the reader to avoid cumbersome floor/ceil-notations.

If $0 \le \lambda < \frac{s}{t}$, the exponent is negative and hence the sum is uniformly bounded by a constant, leading to

$$B \lesssim 2^{(s-t)J} \|u\|_{H^t_{\mathrm{mix}}(\mathbb{T}^2)}.$$

If $\lambda = \frac{s}{t}$, the exponent in the sum is equal to zero such that the sum is proportional to J which yields an additional logarithmic factor, i.e.

$$B \lesssim J 2^{(s-t)J} \|u\|_{H^t_{\mathrm{mix}}(\mathbb{T}^2)}.$$

For the expression C we have

$$\begin{aligned}
\frac{C}{2} &\lesssim \sum_{j_1 = J\frac{1-\lambda}{2-\lambda}}^{\infty} \sum_{j_2=j_1}^{\infty} 2^{(s-t)j_2 - tj_1} \|u\|_{H^t_{\mathrm{mix}}(\mathbb{T}^2)} \\
&\lesssim \sum_{j_1 = J\frac{1-\lambda}{2-\lambda}}^{\infty} 2^{(s-2t)j_1} \underbrace{\sum_{j_2=0}^{\infty} 2^{(s-t)j_2}}_{\lesssim 1} \|u\|_{H^t_{\mathrm{mix}}(\mathbb{T}^2)} \\
&\lesssim 2^{J(s-2t)\frac{1-\lambda}{2-\lambda}} \underbrace{\sum_{j_1=0}^{\infty} 2^{(s-2t)j_1}}_{\lesssim 1} \|u\|_{H^t_{\mathrm{mix}}(\mathbb{T}^2)} \lesssim 2^{J(s-t)} \|u\|_{H^t_{\mathrm{mix}}(\mathbb{T}^2)},
\end{aligned}$$

where we used in the the last step that

$$(2t-s)\frac{1-\lambda}{2-\lambda} \ge t-s$$

whenever $\lambda \in [0, \frac{s}{t}]$.

Putting the estimates of $A = B$ and C together yields the desired result. □

Remark 3.6. We note that the above proof is inherently different from that in [13, 14], which is based on wavelets, since we now cannot exploit norm equivalences any more for our kernel approach. As a consequence, we obtain the logarithmic factor J for the choice $\lambda = \frac{s}{t}$, which is not contained in the error estimate of [13, 14]. Nevertheless, in case of $\lambda \in (0, \frac{s}{t})$, neither the rate of approximation nor the number N of degrees of freedom in $\widehat{\boldsymbol{V}}^{\lambda}_J$ exhibit a logarithmic factor. This means that the convergence rate does not suffer from curse of dimension, i.e., we obtain the same rate as for the univariate kernel interpolation:

$$\big\|u - \widehat{\boldsymbol{Q}}^{\lambda}_J u\big\|_{H^s_{\mathrm{iso}}(\mathbb{T}^2)} \lesssim N^{-(t-s)} \|u\|_{H^t_{\mathrm{mix}}(\mathbb{T}^2)}.$$

Having this result proven, we can easily generalize it to Sobolev spaces of hybrid regularity.

Theorem 3.7 (General convergence)**.** *Assume* $0 \le s_2 \le s_1 \le p$ *and* $0 \le t_1 \le t_2 \le 2p$ *such that we have*[3]

$$H^{2p}_{\mathrm{mix}}(\mathbb{T}^2) \subset H^{s_2,t_2}_{\text{iso-mix}}(\mathbb{T}^2) \subset H^{p}_{\mathrm{mix}}(\mathbb{T}^2) \subset H^{s_1,t_1}_{\text{iso-mix}}(\mathbb{T}^2) \subset L^2(\mathbb{T}^2).$$

Then there holds

$$\big\|u - \widehat{\boldsymbol{Q}}^{\lambda}_J u\big\|_{H^{s_1,t_1}_{\text{iso-mix}}(\mathbb{T}^2)} \lesssim 2^{-J((t_2-t_1)-(s_1-s_2))} \|u\|_{H^{s_2,t_2}_{\text{iso-mix}}(\mathbb{T}^2)}$$

[3]The desired embedding amounts to the inequalities $s_1 + t_1 \le p \le \frac{s_2}{2} + t_2$ and $s_2 + t_2 \le 2p$.

provided that $\lambda \in [0, \frac{s_1-s_2}{t_2-t_1})$*. If* $\lambda = \frac{s_1-s_2}{t_2-t_1}$*, an additional logarithmic factor appears, i.e.*

$$\big\|u - \widehat{\boldsymbol{Q}}_J^{\lambda} u\big\|_{H^{s_1,t_1}_{\text{iso-mix}}(\mathbb{T}^2)} \lesssim J 2^{-J((t_2-t_1)-(s_1-s_2))} \|u\|_{H^{s_2,t_2}_{\text{iso-mix}}(\mathbb{T}^2)}.$$

Proof. In view of Lemma 3.3, we find

$$\begin{aligned}\big\|u - \widehat{\boldsymbol{Q}}_J^{\lambda} u\big\|_{H^{s_1,t_1}_{\text{iso-mix}}(\mathbb{T}^2)} &\le \sum_{\boldsymbol{j}\notin\mathcal{I}_J^\lambda} \|\mathbf{Q}_{\boldsymbol{j}} u\|_{H^{s_1,t_1}_{\text{iso-mix}}(\mathbb{T}^2)} \\ &\lesssim \sum_{\boldsymbol{j}\notin\mathcal{I}_J^\lambda} 2^{(s_1-s_2)\|\boldsymbol{j}\|_\infty} \|\mathbf{Q}_{\boldsymbol{j}} u\|_{H^{s_2,t_1}_{\text{iso-mix}}(\mathbb{T}^2)} \\ &\lesssim \sum_{\boldsymbol{j}\notin\mathcal{I}_J^\lambda} 2^{(s_1-s_2)\|\boldsymbol{j}\|_\infty-(t_2-t_1)\|\boldsymbol{j}\|_1} \|u\|_{H^{s_2,t_2}_{\text{iso-mix}}(\mathbb{T}^2)}.\end{aligned}$$

Proceeding now in complete analogy to the proof of Theorem 3.5, but with $s_1 - s_2$ instead of s and $t_2 - t_1$ instead of t, yields the desired claim. □

Remark 3.8. The condition $s_1 \ge s_2$ is essential for the present result, meaning that we measure the error in a space which has a higher isotropic smoothness than the function to be approximated. If $s_1 = s_2$, we find $\lambda = 0$, such that the interval $[0, \frac{s_1-s_2}{t_2-t_1})$ is empty and we always obtain the logarithmic factor J in the approximation error. We should further mention that the result of Theorem 3.7 could also be shown in [13, 14] for $s_1 < s_2$ due to the norm equivalences of wavelet bases. This is however not possible in our situation using kernel interpolants. Nevertheless, $s_1 < s_2$ also implies $\lambda < 0$ and hence results in associated optimized sparse grids which contain significantly more points than the regular sparse grid. As a consequence, the cost complexity would be higher than (poly-) loglinear in this situation.

4. Extension to higher dimensions

4.1. **Sobolev spaces of hybrid regularity.** In higher dimensions, the Sobolev spaces of dominating mixed derivatives are defined by

$$H^t_{\text{mix}}(\mathbb{T}^d) := \bigotimes_{j=1}^d H^t_{\text{mix}}(\mathbb{T}).$$

This means that $H^t_{\text{mix}}(\mathbb{T}^d)$ contains all periodic functions $f \in L^2(\mathbb{T}^d)$ with bounded derivatives $\|\partial^{\boldsymbol{\alpha}} f\|_{L^2(\mathbb{T}^d)} < \infty$ for all $\|\boldsymbol{\alpha}\|_\infty \le t$. In contrast, the classical, isotropic Sobolev space

$$H^s_{\text{iso}}(\mathbb{T}^d) := \bigcap_{i=1}^d \left[\left(\bigotimes_{j=1}^{i-1} L^2(\mathbb{T})\right) \otimes H^s(\mathbb{T}) \otimes \left(\bigotimes_{j=i+1}^{d} L^2(\mathbb{T})\right)\right]$$

contains all periodic function with bounded derivatives $\|\partial^{\boldsymbol{\alpha}} f\|_{L^2(\mathbb{T}^d)} < \infty$ for all $\|\boldsymbol{\alpha}\|_1 \le s$.

The Sobolev space of hybrid regularity can be defined in analogy to (3.1) as

$$H^{s,t}_{\text{iso-mix}}(\mathbb{T}^d) := \bigcap_{i=1}^d \left[\left(\bigotimes_{j=1}^{i-1} H^t(\mathbb{T})\right) \otimes H^{s+t}(\mathbb{T}) \otimes \left(\bigotimes_{j=i+1}^{d} H^t(\mathbb{T})\right)\right].$$

It consists of all periodic functions $f \in L^2(\mathbb{T}^d)$ with derivatives $\|\partial^{\boldsymbol{\alpha}} f\|_{H^t_{\text{mix}}(\mathbb{T}^d)} < \infty$ bounded in $H^t_{\text{mix}}(\mathbb{T}^d)$ for all $\|\boldsymbol{\alpha}\|_1 \leq s$.

Note that there holds the series of embeddings

$$H^{s+\frac{t}{d}}_{\text{mix}}(\mathbb{T}^d) \subset H^{s+\frac{t}{d}}_{\text{iso}}(\mathbb{T}^d) \subset H^{s,t}_{\text{iso-mix}}(\mathbb{T}^d) \subset H^s_{\text{iso}}(\mathbb{T}^d) \subset H^{\frac{s}{d}}_{\text{mix}}(\mathbb{T}^d) \tag{4.1}$$

for all $s, t \geq 0$ and $d \geq 1$.

4.2. **Kernel interpolation.** We now investigate kernel interpolation in $H^p_{\text{mix}}(\mathbb{T}^d)$. The reproducing kernel $\boldsymbol{\kappa}(\boldsymbol{x}, \boldsymbol{y})$ in $H^p_{\text{mix}}(\mathbb{T}^d)$ is given as the d-fold tensor product of the univariate kernel, which means that

$$\boldsymbol{\kappa}(\boldsymbol{x}, \boldsymbol{y}) = \prod_{k=1}^{d} \kappa(x_k, y_k).$$

We define the projections $\boldsymbol{P_j}$ and $\boldsymbol{Q_j}$ by

$$\boldsymbol{P_j} := P_{j_1} \otimes \cdots \otimes P_{j_d}, \quad \boldsymbol{Q_j} := Q_{j_1} \otimes \cdots \otimes Q_{j_d}.$$

Given a function $u \in H^p_{\text{mix}}(\mathbb{T}^d)$, the kernel interpolant

$$\boldsymbol{P_j} u = \sum_{\boldsymbol{k} \in \boldsymbol{\Delta_j}} u_{\boldsymbol{j},\boldsymbol{k}} \boldsymbol{\kappa}(\boldsymbol{x}, \boldsymbol{x}_{\boldsymbol{j},\boldsymbol{k}})$$

with respect to the *(full) tensor product grid*

$$\boldsymbol{X_j} := \bigtimes_{i=1}^{d} X_{j_i} = \left\{ \mathbf{x}_{\boldsymbol{j},\boldsymbol{k}} = (x_{j_1,k_1}, \ldots, x_{j_d,k_d}) : \boldsymbol{k} \in \boldsymbol{\Delta_j} := \bigtimes_{\ell=1}^{d} \Delta_{j_\ell,k_\ell} \right\}$$

is obtained by solving the system

$$(\boldsymbol{K}_{j_1} \otimes \cdots \otimes \boldsymbol{K}_{j_d}) \boldsymbol{u_j} = \boldsymbol{f_j}, \tag{4.2}$$

where the univariate kernel matrices $\boldsymbol{K}_{j_\ell} = [\kappa(x_{j_\ell,k}, x_{j_\ell,k'})]_{k,k' \in \Delta_{j_\ell}}$ are given by (2.4) and

$$\boldsymbol{u_j} = [u_{\boldsymbol{j},\boldsymbol{k}}]_{\boldsymbol{k} \in \boldsymbol{\Delta_j}}, \quad \boldsymbol{f_j} = [u(\boldsymbol{x}_{\boldsymbol{j},\boldsymbol{k}})]_{\boldsymbol{k} \in \boldsymbol{\Delta_j}}.$$

We again note that the linear system (3.4) of equations can efficiently be solved by using the fast Fourier transform in combination with well-known tensor and matrizication techniques found in e.g. [17].

Obviously, the kernel interpolant is the best approximation of $u \in H^p_{\text{mix}}(\mathbb{T}^d)$ in the subspace

$$\boldsymbol{V_j} = \bigotimes_{i=1}^{d} V_{j_i} = \operatorname{span}\{\kappa(\cdot, \boldsymbol{x}) : \boldsymbol{x} \in \boldsymbol{X_j}\} \subset H^p_{\text{mix}}(\mathbb{T}^d)$$

with respect to the $H^p_{\text{mix}}(\mathbb{T}^d)$-norm.

4.3. **Bernstein and Jackson inequalities.** We find the following straightforward extension of Lemma 3.3 to the d-dimensional setting:

Lemma 4.1 (General inverse estimate)**.** *For $0 \leq s_1 \leq s_2 \leq p$ and $0 \leq t_1 \leq t_2 \leq p$ such that $s_2 + t_2 \leq p$ and $u \in H^p_{\text{mix}}(\mathbb{T}^d)$, there holds*

$$\|\boldsymbol{Q_j} u\|_{H^{s_2,t_2}_{\text{iso-mix}}(\mathbb{T}^d)} \lesssim 2^{(s_2-s_1)\|\mathbf{j}\|_\infty + (t_2-t_1)\|\mathbf{j}\|_1} \|\boldsymbol{Q_j} u\|_{H^{s_1,t_1}_{\text{iso-mix}}(\mathbb{T}^d)}$$

for all $\boldsymbol{j} \in \mathbb{N}_0^d$.

Now, we introduce for $\lambda \in (-\infty, 1)$ the index set

$$\mathcal{I}_J^\lambda = \{\boldsymbol{j} \in \mathbb{N}_0^d : \|\boldsymbol{j}\|_1 - \lambda\|\boldsymbol{j}\|_\infty \le J(1-\lambda)\}.$$

This yields the *optimized sparse grid space*

$$\widehat{\boldsymbol{V}}_J^\lambda := \sum_{\boldsymbol{j}\in\mathcal{I}_J^\lambda} \boldsymbol{V}_{\boldsymbol{j}} = \sum_{\boldsymbol{j}\in\mathcal{I}_J^\lambda} \bigotimes_{i=1}^d V_{j_i},$$

which has the dimension

$$\dim\left(\widehat{\boldsymbol{V}}_J^\lambda\right) \lesssim \begin{cases} 2^J, & \text{if } \lambda > 0,\\ 2^J J^{d-1}, & \text{if } \lambda = 0,\\ 2^{Jd\frac{1-\lambda}{d-\lambda}}, & \text{if } \lambda < 0. \end{cases}$$

The last case is seen as follows: If we are looking for the largest level $\mathbf{j} \in \mathbb{N}_0^d$ with $j_1 = \cdots = j_d =: j$ such that the index $\boldsymbol{j}$ is still contained in $\mathcal{I}_J^\lambda$, we find

$$dj - \lambda j = J(1-\lambda) \Rightarrow j = J\frac{1-\lambda}{d-\lambda}$$

in analogy to (3.10). This is the largest full tensor product space $\boldsymbol{V}_{\boldsymbol{j}} = \bigotimes_{j=1}^d V_j$ that is contained in $\widehat{\boldsymbol{V}}_J^\lambda$. It has

$$\dim(\boldsymbol{V}_{\boldsymbol{j}}) = \dim(V_j)^d = 2^{Jd\frac{1-\lambda}{d-\lambda}}$$

degrees of freedom and gives the bound for $\dim\left(\widehat{\boldsymbol{V}}_J^\lambda\right)$ in the case $\lambda < 0$, compare also [13, 14].

Finally, the generalization of Theorem 3.7 to the d-dimensional setting reads as follows, compare also [13, 14]:

Theorem 4.2 (General convergence)**.** *Assume $0 \le s_2 \le s_1 \le p$ and $0 \le t_1 \le t_2 \le 2p$ such that*[4]

$$H_{\text{mix}}^{2p}(\mathbb{T}^d) \subset H_{\text{iso-mix}}^{s_2,t_2}(\mathbb{T}^d) \subset H_{\text{mix}}^{p}(\mathbb{T}^d) \subset H_{\text{iso-mix}}^{s_1,t_1}(\mathbb{T}^d) \subset L^2(\mathbb{T}^d). \tag{4.3}$$

Then there holds

$$\left\|u - \widehat{\boldsymbol{Q}}_J^\lambda u\right\|_{H_{\text{iso-mix}}^{s_1,t_1}(\mathbb{T}^d)} \lesssim 2^{-J((t_2-t_1)-(s_1-s_2))}\|u\|_{H_{\text{iso-mix}}^{s_2,t_2}(\mathbb{T}^d)}$$

provided that $\lambda \in [0, \frac{s_1-s_2}{t_2-t_1})$. If $\lambda = \frac{s_1-s_2}{t_2-t_1}$, an additional polylogarithmic factor appears, i.e.

$$\left\|u - \widehat{\boldsymbol{Q}}_J^\lambda u\right\|_{H_{\text{iso-mix}}^{s_1,t_1}(\mathbb{T}^d)} \lesssim J^{d-1}2^{-J((t_2-t_1)-(s_1-s_2))}\|u\|_{H_{\text{iso-mix}}^{s_2,t_2}(\mathbb{T}^d)}.$$

Proof. The proof is along the lines of the proof of Theorem 3.5. One has

$$\begin{aligned}\left\|u - \widehat{\boldsymbol{Q}}_J^\lambda u\right\|_{H_{\text{iso-mix}}^{s_1,t_1}(\mathbb{T}^2)} &\le \sum_{\boldsymbol{j}\notin\mathcal{I}_J^\lambda} \|\mathbf{Q}_{\boldsymbol{j}} u\|_{H_{\text{iso-mix}}^{s_1,t_1}(\mathbb{T}^2)}\\ &\lesssim \sum_{\boldsymbol{j}\notin\mathcal{I}_J^\lambda} 2^{(s_1-s_2)\|\boldsymbol{j}\|_\infty}\|\mathbf{Q}_{\boldsymbol{j}} u\|_{H_{\text{iso-mix}}^{s_2,t_1}(\mathbb{T}^2)}\\ &\lesssim \sum_{\boldsymbol{j}\notin\mathcal{I}_J^\lambda} 2^{(s_1-s_2)\|\boldsymbol{j}\|_\infty-(t_2-t_1)\|\boldsymbol{j}\|_1}\|u\|_{H_{\text{iso-mix}}^{s_2,t_2}(\mathbb{T}^2)}.\end{aligned}$$

[4]In view of (4.1), the desired embedding amounts to the inequalities $s_1 + t_1 \le p \le \frac{s_2}{d} + t_2$ and $s_2 + t_2 \le 2p$.

We find

$$\{\boldsymbol{j} \in \mathbb{N}_0^d : \|\boldsymbol{j}\|_1 - \lambda\|\boldsymbol{j}\|_\infty > J(1-\lambda)\} = \bigcup_{k=1}^{d} A_k$$

with

$$A_k := \left\{\boldsymbol{j} \in \mathbb{N}_0^d : j_k = \|\boldsymbol{j}\|_\infty \wedge \sum_{\substack{i=1\\ i\neq k}}^{d} j_i + (1-\lambda)j_k > J(1-\lambda)\right\}.$$

Hence,

$$\big\|u - \widehat{\boldsymbol{Q}}_J^\lambda u\big\|_{H^{s_1,t_1}_{\text{iso-mix}}(\mathbb{T}^2)} \lesssim \sum_{k=1}^{d} \sum_{\boldsymbol{j}\in A_k} 2^{-(t_2-t_1)\sum_{i\neq k} j_i} 2^{((s_1-s_2)-(t_2-t_1))j_k} \|u\|_{H^{s_2,t_2}_{\text{iso-mix}}(\mathbb{T}^2)}.$$

Since there holds

$$A_k \subset B_k := \left\{\boldsymbol{j} \in \mathbb{N}_0^d : \sum_{\substack{i=1\\ i\neq k}}^{d} j_i + (1-\lambda)j_k > J(1-\lambda)\right\},$$

we can further estimate

$$\big\|u - \widehat{\boldsymbol{Q}}_J^\lambda u\big\|_{H^{s_1,t_1}_{\text{iso-mix}}(\mathbb{T}^2)} \lesssim \sum_{k=1}^{d} \sum_{\boldsymbol{j}\in B_k} 2^{-(t_2-t_1)\sum_{i\neq k} j_i} 2^{((s_1-s_2)-(t_2-t_1))j_k} \|u\|_{H^{s_2,t_2}_{\text{iso-mix}}(\mathbb{T}^2)}.$$

The error contributions in each of the simplicial index sets B_k corresponds to the error of a generalized sparse grid with weights $\alpha_i = \frac{1}{1-\lambda}$ for $i \neq k$ and $\alpha_k = 1$, compare [11]. Thus, the k-th error contributions can be estimated by using [11], leading to the desired bound. In particular, the polylogarithmic factor J^{d-1} appears only in the case when the error contributions are identical along the boundary of the index set $\mathcal{I}_J^\lambda$, which is only the case for $\lambda = \frac{s_1-s_2}{t_2-t_1}$. □

4.4. **Computation of the sparse grid kernel interpolant.** We should finally comment on the computation of the kernel interpolant $\widehat{\boldsymbol{Q}}_J^\lambda u$. We may employ the *sparse grid combination technique* as introduced in [16, 29]. To this end, we use the identity

$$\widehat{\boldsymbol{Q}}_J^\lambda = \sum_{\boldsymbol{j}\in\mathcal{I}_J^\lambda} c_{\boldsymbol{j}} \boldsymbol{P}_{\boldsymbol{j}}, \quad \text{where } c_{\boldsymbol{j}} := \sum_{\substack{\boldsymbol{j}'\in\{0,1\}^d:\\ \boldsymbol{j}+\boldsymbol{j}'\in\mathcal{I}_J^\lambda}} (-1)^{\|\boldsymbol{j}'\|_1}, \tag{4.4}$$

compare [6, 7]. Hence, the sought sparse grid kernel interpolant $\widehat{\boldsymbol{Q}}_J^\lambda u$ is composed by the tensor product kernel interpolants $u_{\boldsymbol{j}} := \boldsymbol{P}_{\boldsymbol{j}} u$ from the different full tensor product spaces $\boldsymbol{V}_{\boldsymbol{j}}$ with $c_{\boldsymbol{j}} \neq 0$. Each of the tensor product kernel interpolants $u_{\boldsymbol{j}}$ can now be computed (completely in parallel) by solving the linear system (4.2) of equations. We emphasize that the combination technique does not introduce an additional consistency error for the problem under consideration, see [12] for a proof.

5. Conclusion

In the present article, we have shown that the approximation rate for kernel interpolation in $H^p_{\text{mix}}(\mathbb{T}^d)$ with respect to optimized sparse grids, i.e.

$$\big\|u - \widehat{\boldsymbol{Q}}_J^{\lambda} u\big\|_{H^{s_1,t_1}_{\text{iso-mix}}(\mathbb{T}^d)} \lesssim 2^{-J((t_2-t_1)-(s_1-s_2))}\|u\|_{H^{s_2,t_2}_{\text{iso-mix}}(\mathbb{T}^d)}, \tag{5.1}$$

is dimension independent for the choice $0 < \lambda < \frac{s_1-s_2}{t_2-t_1}$ whenever $0 \le s_2 \le s_1 \le p$ and $0 \le t_1 \le t_2 \le 2p$ such that (4.3) holds and $u \in H^p_{\text{mix}}(\mathbb{T}^d)$ is sufficiently smooth. Nevertheless we like to emphasize that the generic constant which is involved in this error estimate still depends (exponentially) on the dimension d.

The result (5.1) carries over straighforwardly to *quasi-uniform* point sets

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset \mathbb{T}$$

instead of equidistant point sets, where the cardinality of the point sets

$$X_j := \{x_{j,k} : k \in \Delta_j\}$$

satisfies $|\Delta_j| \sim 2^j$. The notion quasi-uniform means in the present context that the *fill distance* satisfies

$$h_j := \sup_{x\in\mathbb{T}} \min_{x_k\in X_j} \|x - x_k\|_2 \lesssim 2^{-j}$$

while the *separation radius* satisfies

$$q_j := \min_{\substack{x_k,x_\ell\in X_j:\\ x_k\neq x_\ell}} \frac{1}{2}\|x_k - x_\ell\|_2 \gtrsim 2^{-j}.$$

Our result applies moreover one-to-one to general product domains and the non-periodic case as described in [12], if the *doubling trick* from [26, 28] is applicable. Thanks to this trick we are able to exploit extra smoothness if present for the function to be approximated. Nonetheless, if the doubling trick does not apply, we still have an estimate of the form

$$\big\|u - \widehat{\boldsymbol{Q}}_J^{\lambda} u\big\|_{H^{s,t}_{\text{iso-mix}}(\mathbb{T}^d)} \lesssim 2^{-J((p-t)-s)}\|u\|_{H^{p}_{\text{mix}}(\mathbb{T}^d)}$$

whenever $0 < s$, $0 \le t$, and $s + \frac{t}{d} \le p$. Hence, in general, no logarithm appears in the approximation rate for kernel interpolation if we measure the error with respect to an isotropic Sobolev space being different from $L^2(\mathbb{T}^d)$ for the choice $0 < \lambda < \frac{s}{t}$.

In case of the Schrödinger equation, one has the product of three-dimensional one-particle spaces instead of just one-dimensional spaces like in our article here and $s_1 = 1$, $t_1 = 0$, and $s_2 = 1$, while $\frac{1}{2} \le t_2 \le 1$ depends on the particular symmetry behaviour of the wavefunctions, see [24, 36] for the details. This means that the usual sparse grid space (i.e., the space $\widehat{\boldsymbol{V}}_J^{\lambda}$ with $\lambda = 0$, now on three-dimensional particle spaces) would be optimal, which results in (poly-) logarithmic factors in the number of degrees of freedom as well as in the rate of approximation. Note that the antisymmetry of the wavefunctions can be built into the kernel interpolant in accordance with [23].

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