A DOMAIN DECOMPOSITION METHOD USING SPARSE GRIDS

MICHAEL GRIEBEL

ABSTRACT. We present preconditioners for linear systems arising from sparse and full grid discretizations of PDE's and discuss their application to the domain decomposition method.

1. FULL AND SPARSE GRIDS, FULL AND SPARSE GRID PROBLEMS

We consider a partial differential equation

$$Lu = f$$

in the unit square $\bar{\Omega} = [0,1]^2 \subset \mathbb{R}^2$ with a linear, elliptic operator L of second order and appropriate boundary conditions. For reasons of simplicity, we restrict ourselves to the case of homogeneous boundary conditions.

For the discretization of the problem we use a grid $\Omega_{i,j}$ with grid sizes $h_x = 2^{-i}$, $h_y = 2^{-j}$ in x- and y-direction. Associated is the space of piecewise bilinear functions $V_{i,j}$ vanishing on the boundary. The dimension of $V_{i,j}$ is $(2^i - 1)(2^j - 1) = O(2^{i+j})$. The Galerkin approach now leads to the linear system $L_{i,j}\, u_{i,j} = f_{i,j}$ that can be solved effectively for example by a multigrid method. Usually, an equidistant grid $\Omega_{k,k}$ with associated space $V_{k,k}$ of dimension of $O(2^{2k})$ is used. Note that this space can be decomposed as

$$V_{k,k} = \sum_{i=1}^{k} \sum_{j=1}^{k} V_{i,j}\,.$$

Alternatively, we consider the so called sparse grid space

$$V_k^S = \sum_{i=1}^{k} \sum_{j=1}^{k+1-i} V_{i,j}\,.$$

with dimension $n_k^S = (k-1)\cdot 2^k - 1 = O(k \cdot 2^k)$. It is associated to the sparse grid Ω_k^S. See Figure 1. Note that for the representation of a function of V_k^S a product type hierarchical basis is advisable. For details, see [12], [13]. The Galerkin approach now leads to the associated sparse grid problem

$$L_k^S u_k^S = f_k^S\,. \tag{1.1}$$

1991 *Mathematics Subject Classification.* 65F10, 65N20, 65N30.

Key words and phrases. sparse grids, combination method, preconditioning.

partially supported by DFG-Sonderforschungsbereich SFB 0342

Note that L_k^S is more densely populated than $L_{k,k}$. However, based on the hierarchical basis mentioned above, it exists a product type representation of L_k^S allowing a matrix vector multiplication in $O(n_k^S)$ operations.

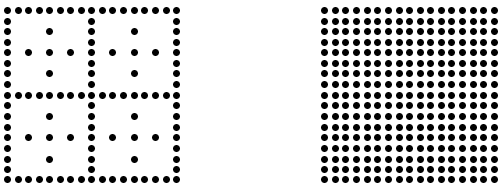

FIGURE 1. The sparse grid $\Omega_{4,4}^s$ and the associated full grid $\Omega_{4,4}$.

While the full grid problem contains $O(4^k)$ unknowns, the sparse grid problem (1.1) contains only $O(k \cdot 2^k)$ unknowns. The accuracy of the sparse grid solution u_k^S, however, is practically the same as for the full grid solution $u_{k,k}$. In [1] it is shown that both $|u - u_{k,k}|_E$ and $|u - u_k^S|_E$ are of the order $O(2^{-k})$ provided that the solution u is sufficiently smooth to fulfil

$$|\frac{\partial u^4}{\partial x^2 \partial y^2}| \leq C. \tag{1.2}$$

Thus, it is often cheaper to solve the sparse grid problem instead of the full grid problem. The solution u_k^S of (1.1) can be obtained efficiently by special multigrid methods citeBUN92, [2], [4], that involve in general quite complicated data structures and algorithms. In the following, we present a simple preconditioner for (1.1).

2. THE COMBINATION METHOD AS A SPARSE GRID PRECONDITIONER

Consider now the so-called combination solution

$$u_{n,n}^C = \sum_{i=1}^{k} u_{i,j} - \sum_{i=1}^{k-1} u_{i,j-1} \tag{2.3}$$

with $j := k+1-i$, that has been introduced in [6]. To obtain $u_{k,k}^C$, we have to solve k different problems $L_{i,j} u_{i,j} = f_{i,j}, i = 1,..,k, j = k+1-i$, each with about 2^k unknowns, and $k-1$ different problems $L_{i,j} u_{i,j} = f_{i,j}, i = 1,..,k-1, j = k-i$, each with about 2^{k-1} unknowns, and combine their bilinearly interpolated solutions. This gives an approximate solution of the sparse grid problem defined on $\Omega_{n,n}^s$. This method is illustrated in Figure 2.

Note that the solution of the different problems arising in the combination method can be computed independently. Thus the parallelization of the method is straightforward. For results, see [3] or [10]. The solution of each problem is computed on standard grids. Therefore, the complicated data structures and algorithms necessary for the sparse grid approach are avoided and we can use standard codes. A multigrid method with semi-refinement to cope with the grid distorsion is recommended.

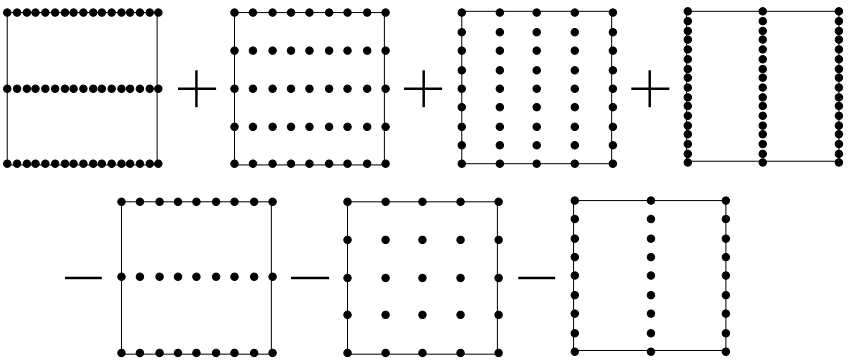

FIGURE 2. The linear combination of grids $\Omega_{i,j}$ and solutions $u_{i,j}$ with $i = 1,..k, j = k+1-i$ and $i = 1,..,k-1, j = k-i, k = 4$.

The combination method gives raise to the preconditioner

$$(B^S_{k,k})^{-1} = \sum_{i=1}^{k} P^{k,k,S}_{i,k+1-i} L^{-1}_{i,k+1-i} R^{i,k+1-i}_{k,k,S} - \sum_{i=1}^{k} P^{k,k,S}_{i,k-i} L^{-1}_{i,k-i} R^{i,k-i}_{k,k,S}. \tag{2.4}$$

for the sparse grid problem (1.1). Here, $P^{k,k,S}_{i,j} : V_{i,j} \to V^S_{k,k}$ denotes the interpolation operator and $R^{i,j}_{k,k,S} : V^S_{k,k} \to V_{i,j}$ denotes the restriction operator defined as the adjoint of $P^{k,k,S}_{i,j}$, $R^{i,j}_{k,k,S} = (P^{k,k,S}_{i,j})^T$.

It is easy to see that $u^C_{k,k} = (B^S_{k,k})^{-1} f^S_{k,k}$. Note that the combination solution $u^C_{k,k}$ is not equal to $u^S_{k,k}$ but a very good approximation to it. It can be shown that

$$\kappa((B^S_{k,k})^{-1} L^S_{k,k}) = O(1) \tag{2.5}$$

and by using the combination method as preconditioner within an appropriate iterative method like a conjugate gradient or a defect correction iteration, we obtain a simple algorithm that needs a number of iterations independent of k to solve the sparse grid problem to a prescribed accuracy.

However, since the discrete solution $u^S_{k,k}$ differs from u by the discretization error anyway, it makes no sense to solve the sparse grid problem to machine accuracy. It is sufficient to perform only as many iterations as necessary to reach the discretization error. To demonstrate this and the property (2.5) we considered the model problem $\Delta u = f$ with exact solution $x * (1-x) * cos(x * \pi/2) * y * (1-y) * cos(y * \pi/2)$ and Dirichlet boundary conditions. We performed conjugate gradient iterations for (1.1) preconditioned by (2.4). The convergence history of the iteration is shown for the cases $k = 3, 4, .., 8$ in Figure 3 (left). We clearly see that the convergence rate and thus the condition number of $(B^S_{k,k})^{-1} L^S_{k,k}$ is independent of k.

In the right part of Figure 3 we show a zoom of the convergence history for the first two iteration steps. Additionally, we indicate where the discretization error (vertical lines) is reached. We see that the discretization error accuracy is already gained after only one iteration step, where just the combination solution $u^C_{k,k}$ is produced. Consequently, for any u that fulfils the requirement (1.2), the error of the combination solution lies within the discretization error accuracy and only one iteration step is necessary.

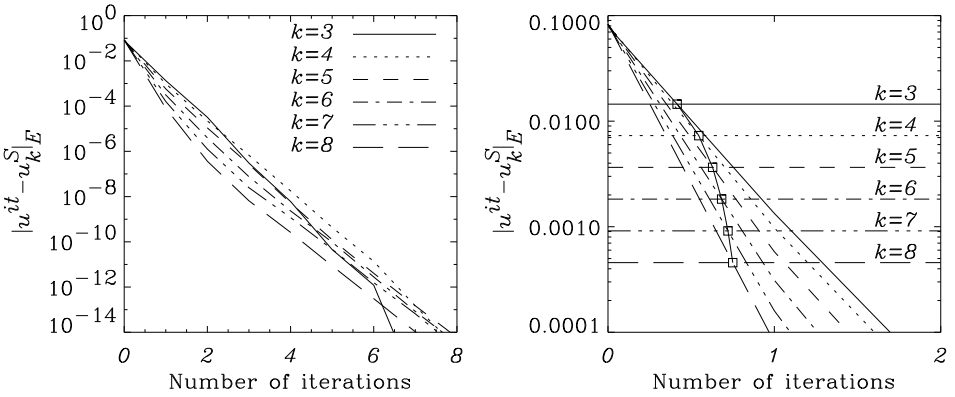

FIGURE 3. Convergence history of CG for preconditioner (2.4).

3. THE COMBINATION METHOD USED IN A FULL GRID PRECONDITIONER

Now, we turn to the problem discretized on the full grid $\Omega_{k,k}$

$$L_{k,k}u_{k,k} = f_{k,k}. \tag{3.6}$$

Let $P_{i,j}^{k,k} : V_{i,j} \to V_{k,k}$ be the interpolation operator and let $R_{k,k}^{i,j} : V_{k,k} \to V_{i,j}$ be the restriction operator defined as the adjoint of $P_{i,j}^{k,k}$, $R_{k,k}^{i,j} = (P_{i,j}^{k,k})^T$. The preconditioner associated to the combination method can now be written as

$$B_{k,k}^{-1} = \sum_{i=1}^{k} P_{i,k+1-i}^{k,k} L_{i,k+1-i}^{-1} R_{k,k}^{i,k+1-i} - \sum_{i=1}^{k} P_{i,k-i}^{k,k} L_{i,k-i}^{-1} R_{k,k}^{i,k-i}. \tag{3.7}$$

However, only functions of the sparse grid subspace $V_{k,k}^S$ are affected by $B_{k,k}^{-1}$ and it is easy to see that $B_{k,k}^{-1}$ is of deficient rank. An iteration for the full grid problem preconditioned by $B_{k,k}$ only would converge to a fix point which is contained in the subspace $V_{k,k}^S$. However, the solution $u_{k,k} \in V_{k,k}$ is never reached. In the multigrid context, $B_{k,k}^{-1}$ can be considered as a certain sort of multivariate additive coarse grid correction similar to [8].

Therefore, we consider for the full grid problem the following additive preconditioner

$$C_{k,k}^{-1} = B_{k,k}^{-1} + \omega \cdot diag(L_{k,k})^{-1} \tag{3.8}$$

that exhibits full rank. It incorporates $(B_{k,k}^S)^{-1}$ on the subspace $V_{k,k}^S$ and a Jacobi type preconditioner on the full space $V_{k,k}$. Here, ω is a properly chosen damping parameter (e.g. $\omega = 1.0$).

Let $\Phi_{k,k}^{GS}$ denote the Gauss-Seidel iteration operator on grid $\Omega_{k,k}$. Additionally, we consider the multiplicative iteration operator

$$M_{k,k}^{-1} = \Phi_{k,k}^{GS} \cdot (I_{k,k} - B_{k,k}^{-1} L_{k,k}) \tag{3.9}$$

that consists of the multivariate additive coarse grid correction induced by $B_{k,k}^{-1}$ followed by one Gauss-Seidel smoothing step on the fine grid $\Omega_{k,k}$.

It turns out that the condition number is not more optimal. For properly chosen ω

$$\kappa(C_{k,k}^{-1} \cdot L_{k,k}) = O(k^{\alpha}) \tag{3.10}$$

holds where α was determined experimentally to be between 3 and 4. Analogously the asymptotic convergence rate $\rho(M_{k,k}^{-1} \cdot L_{k,k})$ is dependent on k. This behavior can be explained by means of Fourier analysis. Compare also [11]. For our previous model problem, the convergence history of the CG-iteration with preconditioner (3.8) is shown in figure 4. For the MG-like iteration with operator (3.9) analogous results have been obtained.

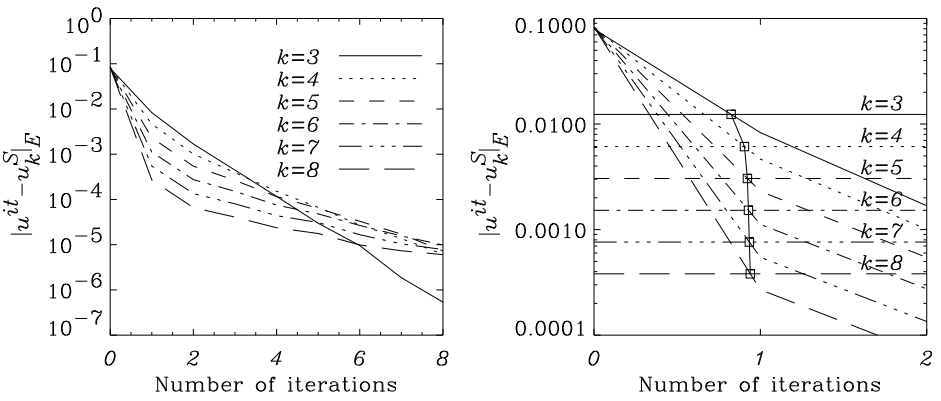

FIGURE 4. Convergence history of CG for preconditioner (3.8).

Note however, that the number of iteration steps now necessary to reach discretization error accuracy is independent of k. We see in Figure 4 (right) that just one iteration step is sufficient for our purpose. With respect to the L_2-norm about two or three iteration steps are necessary.

Thus, there is no use looking only for asymptotically optimal $O(1)$ preconditioners if just discretization error accuracy is needed. We have seen that there exist preconditioners with suboptimal condition number but with associated iterative methods where the number of iterations necessary to reach discretization error accuracy is independent of the mesh size.

4. THE COMBINATION METHOD AND THE DOMAIN DECOMPOSITION METHOD

To combine the combination method preconditioner and the domain decomposition method, basically two approaches exist.

The first one is straightforward. We apply the domain decomposition method (outer loop) using quadrilateral domains and use (3.8) or (3.9) for the solution of each subdomain problem (inner loop). To gain the advantages of the sparse grid approach within the domain decomposition method, we can solve instead of the usual full grid problem only the sparse grid problem on each subdomain e.g. by CG and (2.4). In this way we use just a certain sparse grid for the overall problem.

Alternatively, the combination method can be applied globally (outer loop). See also figure 5 for a simple example involving three blocks. Then, each arising subproblem can be solved by the domain decomposition technique (inner loop).

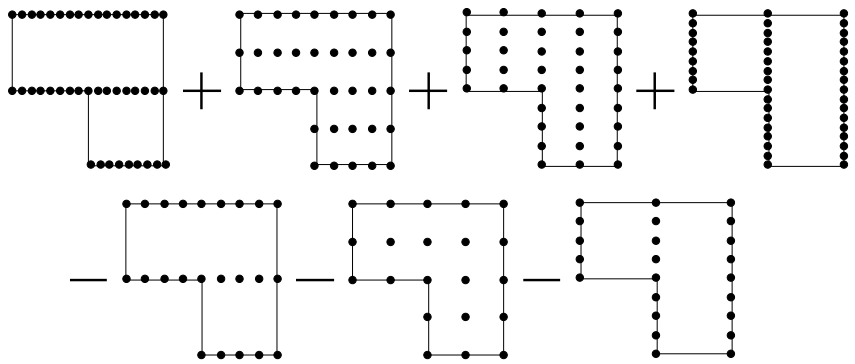

FIGURE 5. The linear combination of grids $\Omega_{i,j}$ and solutions $u_{i,j}$ with $i = 1, ..k, j = k + 1 - i$ and $i = 1, .., k - 1, j = k - i, k = 4$ for the L-shaped domain.

Note that the solution of the different problems arising in the combination method can be computed independently. Thus the parallelization of the method is straightforward. For results, see [6] or [10].

In both cases the parallelization possibilities both of the domain decomposition method and of the combination method can be used simultaneously.

To show at last that the second approach involving the combination technique in the outer loop works well also for more general problems, we studied the Navier-Stokes equations with complicated boundaries. We consider the example of laminar flow over the skyline of Munich (2D idealization) with Reynolds number $Re = 500$. The discretization involves 33 quadrilateral subdomains with curved boundaries and graded meshes to resolve the effects caused by singularities due to reentrant corners. For each block $k = 6$ is chosen.

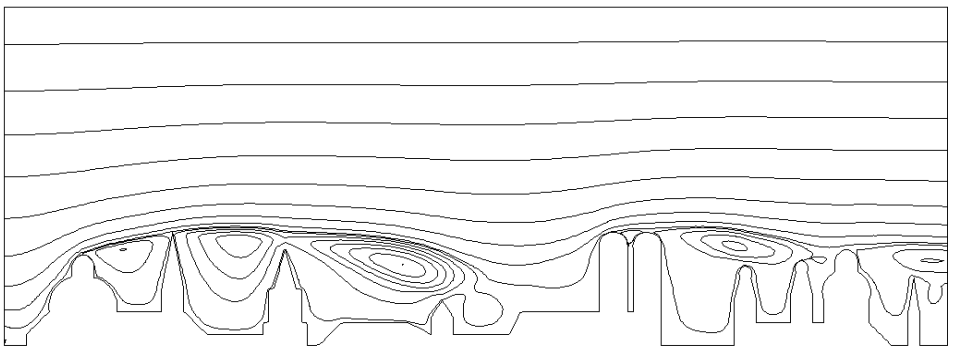

FIGURE 6. Laminar flow over the skyline of Munich, $Re = 500$.

Figure 6 shows the contour lines of the stream function computed for the sparse grid discretization by the combination method. Accuracy experiments with suc-

cessive finer grids showed that the same order of accuracy was obtained with this method as for the solution obtained for the full grid approach. Further experiments and results can be found in [7].

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Institut für Informatik, Technische Universität München, Arcisstrasse 21, D-8000 München 2

E-mail address: griebel@informatik.tu-muenchen.de