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Two Proofs of Convergence for the Combination Technique for the Efficient Solution of Sparse Grid Problems

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ABSTRACT. For a simple model problem — the Laplace equation on the unit square with a Dirichlet boundary function vanishing for $x = 0$, $x = 1$, and $y = 1$, and equaling some suitable $g(x)$ for $y = 0$ — we present a proof of convergence for the combination technique, a modern, efficient, and easily parallelizable sparse grid solver for elliptic partial differential equations that recently gained importance in fields of applications like computational fluid dynamics. For full square grids with meshwidth h and $O(h^{-2})$ grid points, the order $O(h^2)$ of the discretization error using finite differences was shown in [**5**], if $g(x) \in C^2[0,1]$. In this paper, we show that the finite difference discretization error of the solution produced by the combination technique on a sparse grid with only $O\left((h^{-1} \cdot log_2(h^{-1})\right)$ grid points is of the order $O\left(h^2 \cdot log_2(h^{-1})\right)$, if the Fourier coefficients b_k of $\tilde{g}$, the 2-periodic and 0-symmetric extension of g, fulfill $|b_k| \leq c_g \cdot k^{-3-\varepsilon}$ for some arbitrary small positive ε. If $0 < \varepsilon \leq 1$, this is vaild for $g \in C^4[0,1]$ and $g(0) = g(1) = g''(0) = g''(1) = 0$, e.g.. A simple transformation even shows that $g \in C^4[0,1]$ is sufficient. We present results of numerical experiments with functions g of varying smoothness.

1. Introduction

Since their presentation in 1990 [**7**], sparse grids have turned out to be a very interesting tool for the efficient solution of elliptic boundary value problems. Besides the implementation of a hierarchical finite element algorithm on sparse grids [**1**], it has been first of all the combination technique [**4**] that attained attention in the sparse grid context. One of the main advantages of the combination technique stems from the properties of sparse grids [**1**]: In comparison to the standard full grid approach, the number of grid points can be reduced significantly from $O(N^d)$ to $O(N \cdot (\log_2(N))^{d-1})$ in the d-dimensional case, whereas the accuracy of the calculated approximation to the solution is only slightly deteriorated from

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$O(N^{-2})$ (see [**5**]) to $O(N^{-2} \cdot (\log_2(N))^{d-1})$. Another advantage has to be seen in the simplicity of the combination concept, its inherent parallel structure, and its framework property which allows the integration of existing PDE solvers. Up to now, the combination technique has been applied to several types of elliptic PDEs including problems originating from computational fluid dynamics. Its excellent parallelization properties have been verified on different parallel architectures and on workstation networks (see the references in [**3**]).

For a proof of the approximation properties mentioned above, in the 2D case the existence of an error splitting of the type

$$(1)\quad u(x,y) - u_{h_x,h_y}(x,y) = C_1(x,y,h_x)h_x^2 + C_2(x,y,h_y)h_y^2 + C(x,y,h_x,h_y)h_x^2h_y^2$$

with $|C_1(x,y,h_x)|$, $|C_2(x,y,h_y)|$, and $|C(x,y,h_x,h_y)|$ bounded by some positive $B(x,y)$ for all meshwidths h_x and h_y has to be shown [**4**]. Here, $u(x,y)$ is the exact solution of the given boundary value problem, and $u_{h_x,h_y}(x,y)$ denotes the solution on the rectangular full grid with meshwidths h_x and h_y resulting from a finite difference discretization.

We present two proofs for the existence of such an error splitting in the case of Laplace's equation on the unit square, if the Dirichlet boundary function satisfies certain smoothness requirements. Both times, we first split the discretization error $u(x,y) - u_{h_x,h_y}(x,y)$ into three error terms also involving the solutions $u_{h_x,0}$ and u_{0,h_y} of the Laplacian discretized in only one direction. Then, the first approach represents these error terms (differences) as integrals over differential quotients and, thus, reduces statements on $u(x,y) - u_{h_x,h_y}(x,y)$ to statements on some partial derivatives of $u(x,y)$ and $u_{h_x,h_y}(x,y)$. The second proof works in a more straightforward way using the mean value theorem to get derivatives instead of differences. Here, we just outline the essential steps. For details, see [**2**] and [**3**].

2. The Combination Technique

Let L be an elliptic operator of second order. Consider the partial differential equation $Lu = f$ on the unit square $\Omega =]0,1[^2$ with appropriate boundary conditions. Furthermore, let $G_{i,j}$ be the rectangular grid on $\bar{\Omega}$ with meshwidths $h_x := 2^{-i}$ in x- and $h_y := 2^{-j}$ in y-direction ($i, j \in \mathbb{N}$). In [**4**], a technique has been introduced which combines the solutions u_{h_x,h_y} of the discrete problems $L_{h_x,h_y}u_{h_x,h_y} = f_{h_x,h_y}$ associated to $Lu = f$ on different rectangular grids $G_{i,j}$:

$$(2)\qquad u_n^c := \sum_{i+j=n+1} u_{2^{-i},2^{-j}} - \sum_{i+j=n} u_{2^{-i},2^{-j}}.$$

The resulting solution u_n^c is given on the sparse grid $\tilde{G}_{n,n}$ with $O(2^n n)$ grid points instead of the usual full grid $G_{n,n}$ with $O(4^n)$ points. Figure 1 shows the sparse grids $\tilde{G}_{3,3}$ and $\tilde{G}_{4,4}$. For a detailed introduction to sparse grids, see [**7**] and [**1**]. Note that we have to solve n different problems with $O(2^{n+1})$ grid points ($i+j = n+1$) and $n-1$ different problems with $O(2^n)$ grid points ($i+j = n$). These $2n-1$ problems are independent from one another and can be solved in parallel.

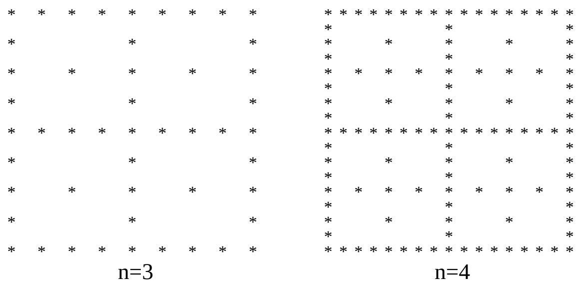

FIGURE 1. Sparse grids $\tilde{G}_{3,3}$ and $\tilde{G}_{4,4}$.

Finally, their bilinearly interpolated solutions have to be combined according to (2).

Assuming that, for each inner point $(x,y) \in \Omega$, the pointwise error of the solution $u_{h_x,h_y} = u_{2^{-i},2^{-j}}$ obtained on the rectangular grid $G_{i,j}$ is of the form (1), it has been shown in [**4**] that the error $e_n^c(x,y)$ of the combined solution $u_n^c(x,y)$ fulfills

$$(3) \qquad |e_n^c(x,y)| \;:=\; |u(x,y) - u_n^c(x,y)| \;\le\; B(x,y) \cdot h_n^2 \cdot \Big(1 + \frac{5}{4} log_2(h_n^{-1})\Big),$$

where $h_n = 2^{-n}$. In this paper, we deal with the question, which smoothness requirements have to be fulfilled by the Dirichlet boundary function in order to get a behaviour of the error $u(x,y) - u_{h_x,h_y}(x,y)$ on $G_{i,j}$ as indicated in (1).

For the remainder, we concentrate on the Laplace equation

$$(4) \qquad \Delta u(x,y) = 0, \;\; (x,y) \in \Omega,$$

with the Dirichlet boundary condition

$$(5) \qquad u(x,y) = \begin{cases} g(x) : (x,y) \in \delta\bar{\Omega}, y = 0 \\ \;\;0 \;\;: (x,y) \in \delta\bar{\Omega}, y > 0. \end{cases}$$

Throughout the paper, we assume that the Fourier coefficients b_k of $\tilde{g}$, the 2-periodic and 0-symmetric extension of g, fulfill

$$(6) \qquad |b_k| \;:=\; \left|\int_{-1}^{1} \tilde{g}(x)\sin(k\pi x)dx\right| \;=\; \left|2\int_0^1 g(x)\sin(k\pi x)dx\right| \;\le\; \frac{c_g}{k^{3+\varepsilon}}$$

for some arbitrary small positive ε and a constant c_g depending only on g. For $0 < \varepsilon \le 1$, e.g., it is a well-known fact from Fourier theory that (6) is valid if $\tilde{g} \in C^4(\mathbb{R})$ or if $g \in C^4[0,1]$ and $g(0) = g(1) = g''(0) = g''(1) = 0$.

3. Explicit Solutions and Error Splitting

In this section, we are looking for explicit representations of the solutions u, u_{h_x,h_y}, $u_{h_x,0}$, and u_{0,h_y} of the Laplace equation (4) and the following three associated discretized problems with meshwidths $h_x = 2^{-i}$ and $h_y = 2^{-j}$:

$$
\begin{aligned}
0 \; &= \; \frac{u_{h_x,h_y}(x-h_x,y) - 2u_{h_x,h_y}(x,y) + u_{h_x,h_y}(x+h_x,y)}{h_x^2} \\
&+ \; \frac{u_{h_x,h_y}(x,y-h_y) - 2u_{h_x,h_y}(x,y) + u_{h_x,h_y}(x,y+h_y)}{h_y^2},
\end{aligned} \tag{7}
$$

$$
0 \; = \; \frac{u_{h_x,0}(x-h_x,y) - 2u_{h_x,0}(x,y) + u_{h_x,0}(x+h_x,y)}{h_x^2} + \frac{\partial^2 u_{h_x,0}(x,y)}{\partial y^2}, \tag{8}
$$

$$
0 \; = \; \frac{\partial^2 u_{0,h_y}(x,y)}{\partial x^2} + \frac{u_{0,h_y}(x,y-h_y) - 2u_{0,h_y}(x,y) + u_{0,h_y}(x,y+h_y)}{h_y^2}. \tag{9}
$$

Thus, u_{h_x,h_y} solves the discrete problem resulting from the use of central finite differences in x- and y-direction. Accordingly, $u_{h_x,0}$ and u_{0,h_y} denote the solutions of the problems discretized only in x- or in y-direction, respectively. Defining

$$
T(t,y) \; := \; \frac{\sinh(t\pi(1-y))}{\sinh(t\pi)}, \tag{10}
$$

we know from [**6**] that the solution u of (4) fulfilling (5) can be written as

$$
u(x,y) := \sum_{k=1}^{\infty} b_k \sin(k\pi x) T(k,y). \tag{11}
$$

As shown in [**3**], the three discrete problems (7)–(9) are solved by

$$
u_{h_x,h_y}(x,y) \; := \; \sum_{k=1}^{\infty} b_k \sin(k\pi x) T(\mu_k,y), \tag{12}
$$

$$
u_{h_x,0}(x,y) \; := \; \sum_{k=1}^{\infty} b_k \sin(k\pi x) T(\nu_k,y), \tag{13}
$$

$$
u_{0,h_y}(x,y) \; := \; \sum_{k=1}^{\infty} b_k \sin(k\pi x) T(\lambda_k,y), \tag{14}
$$

where

$$
\begin{aligned}
\mu_k(h_x,h_y) \; &:= \; \frac{2}{\pi h_y} \cdot \operatorname{arsinh}\left(\frac{h_y}{h_x} \cdot \sin\left(\frac{k\pi h_x}{2} \right) \right), \\
\nu_k(h_x) \; := \; \lim_{h_y \to 0} \mu_k(h_x,h_y), & \qquad \lambda_k(h_y) \; := \; \lim_{h_x \to 0} \mu_k(h_x,h_y).
\end{aligned} \tag{15}
$$

In each sum (12), (13), and (14), $\sin(k\pi x)T(.,y)$ solves the corresponding problem (7), (8), and (9), respectively. The series converge, because $T(t,y) \le 1$ and because of our smoothness assumptions (6). Furthermore, u_{h_x,h_y}, $u_{h_x,0}$, and u_{0,h_y} fulfill all he same boundary condition (5) as u does. This results from the fact that we always use the continuous Fourier coefficients b_k.

Now, we use the explicit representations of u, u_{h_x,h_y}, $u_{h_x,0}$, and u_{0,h_y} to split the discretization error $u_{h_x,h_y} - u$ into three terms:

$$
u_{h_x,h_y} - u \; = \; \Gamma^{(1)}_{h_x} + \Gamma^{(2)}_{h_y} + \Gamma_{h_x,h_y}, \tag{16}
$$

where

$$\Gamma^{(1)}_{h_x} := u_{h_x,0} - u, \qquad \Gamma^{(2)}_{h_y} := u_{0,h_y} - u, \qquad \Gamma_{h_x,h_y} := u_{h_x,h_y} - u_{h_x,0} - u_{0,h_y} + u. \tag{17}$$

We have to learn under which circumstances

$$\Gamma^{(1)}_{h_x} = h_x^2 \cdot C_1(x,y,h_x), \qquad \Gamma^{(2)}_{h_y} = h_y^2 \cdot C_2(x,y,h_y), \qquad \Gamma_{h_x,h_y} = h_x^2 h_y^2 \cdot C(x,y,h_x,h_y) \tag{18}$$

hold, where $|C_1|$, $|C_2|$, and $|C|$ are bounded from above by some $B(x,y)$.

4. Order of the Error Terms

We restrict ourselves to Γ_{h_x,h_y}. Looking at the explicit representations (11)–(14) of u, $u_{h_x,0}$, u_{0,h_y}, and u_{h_x,h_y}, we see that Γ_{h_x,h_y} can be written as a series, too:

$$\Gamma_{h_x,h_y} = \sum_{k=1}^{\infty} b_k \sin(k\pi x) \cdot \Big(T(\mu_k, y) - T(\nu_k, y) - T(\lambda_k, y) + T(k, y) \Big). \tag{19}$$

Here, the crucial task is to transform the differences in (19) to derivatives, since the latter ones are easier to deal with. We study two possible ways of doing so.

For the first approach, we note that $\Gamma_{h_x,h_y} = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \gamma_{h_x \cdot 2^{-l}, h_y \cdot 2^{-m}}$, where $\gamma_{h_x,h_y} = u_{h_x,h_y} - u_{h_x,h_y/2} - u_{h_x/2,h_y} + u_{h_x/2,h_y/2}$. Then, introducing $v(x,y,s,t) := u_{2^{-s},2^{-t}}(x,y)$ for arbitrary $s,t \geq 1$, we get

$$\gamma_{h_x,h_y} = \gamma_{2^{-i},2^{-j}} = \int_i^{i+1} \int_j^{j+1} \frac{\partial^2 v(x,y,\sigma,\tau)}{\partial\sigma\partial\tau} d\sigma d\tau. \tag{20}$$

In a last step, we have to find estimates for the partial derivative $\frac{\partial^2 v}{\partial\sigma\partial\tau}$. This can be done profiting from the help of tools of symbolic computation (see [**2**] for details).

The second proof starts with studying properties of T, ν_k, λ_k, and μ_k introduced in (10) and (15). As a result, applying the mean value theorem twice, we get

$$\begin{aligned} & T(\mu_k, y) - T(\nu_k, y) - T(\lambda_k, y) + T(k, y) \\ = \; & \frac{\partial}{\partial h_y} \Big(T_t\big(F(\xi_k, h_y), y\big) \cdot F_t(\xi_k, h_y) \Big) \Big|_{\alpha h_y} \cdot h_y \cdot (\nu_k - k), \end{aligned} \tag{21}$$

where $F(t,h) := \operatorname{arsinh}(th\frac{\pi}{2})/(h\frac{\pi}{2})$, $\xi_k \in \,]\nu_k, k[$, and $\alpha \in \,]0,1[$. Finally, we have to find bounds for the occurring partial derivatives of T and F (see [**3**] for details).

5. Numerical Experiments

We present some numerical results for the Laplacian on $\Omega =]0,1[^2$ with Dirichlet boundary conditions and the solutions $u^{(\alpha)}(x,y) := Im\left((x - \frac{1}{2} + iy)^\alpha\right)$ and $u(x,y) := \sin(\pi y) \cdot \sinh(\pi(1-x))/\sinh(\pi)$. Here, α indicates the smoothness of $u^{(\alpha)}$ in the critical point $(\frac{1}{2}, 0)$. If, e.g., $\alpha \geq 4$, then $u^{(\alpha)}(x,0) \in C^4([0,1])$. Note that these examples can be reduced to the situation of (4), (5). For the discretization on the full grids $G_{i,j}$, we used finite differences as indicated in (7). The

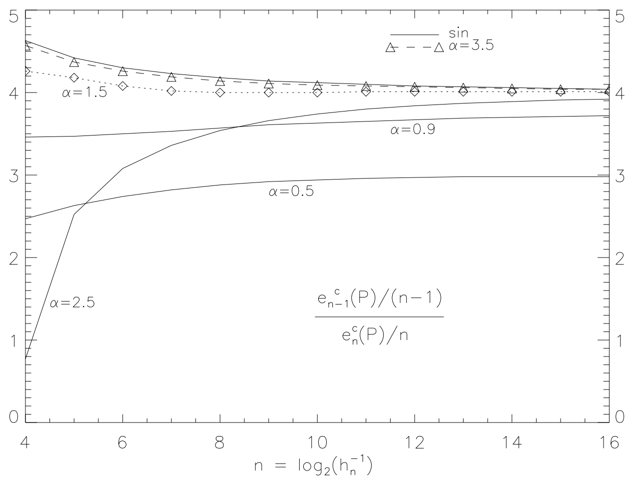

FIGURE 2. Decrease in $e_n^c(P)/n$ proceeding from level $n-1$ to level n.

resulting systems were solved with the help of the NAG library routine D03EDF written by P. Wesseling (see [**3**] for details). Figure 2 shows the decrease in $e_n^c(P)/n$ proceeding from level $n-1$ to level n for our examples. Here, $e_n^c(x,y) := u(x,y) - u_n^c(x,y)$ denotes the error of the combined solution, and $P := (\frac{1}{4}, \frac{1}{4})$. We see the $O(h_n^2 \log_2(h_n^{-1}))$-behaviour for $\alpha \in \{1.5, 2.5, 3.5\}$ and for u and a worse convergence if $\alpha < 1$.

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