HYPERBOLIC CROSS APPROXIMATION OF INTEGRAL OPERATORS WITH SMOOTH KERNEL*

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Abstract. This paper is concerned with the construction and use of trigonometric approximation spaces for the approximate evaluation of integral operators with smooth kernels. The smoothness classes we consider are mixtures of classes of functions of dominating mixed derivative. We define a scale of nested approximation spaces for the approximation of the kernel that includes the standard full grid spaces as well as the spaces related to hyperbolic cross points. We present theoretical results on the approximation power of these spaces and show under which circumstances these approximation spaces lead to algorithms that break the curse of dimensionality. Blending schemes for these new approximation spaces allow the use of simple data structures and the direct application of fast hierarchical methods such as multilevel methods and fast Fourier transforms. Numerical examples illustrate the theoretical findings. The results carry over to other multiscale tensor-product basis functions (specifically to biorthogonal wavelets) and to other tensor-product domains.

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1. Introduction. Despite the rapid development of computer technology, many problems in scientific computing remain intractable to a direct numerical simulation due to storage requirements and numerical complexity. Especially problems in higher dimension remain well out of the range of todays largest computers even after the use of parallelization. Specifically, variational problems with solution in the standard Sobolev spaces $\mathcal{H}^t$ [A75] are well out of reach in higher dimensions. This is often called the "curse of dimensionality" [Wer91]. The situation often looks different when only problems with solution of higher regularity such as functions with dominating mixed derivatives are considered. Under this assumption, subspaces with relatively large dimensions that contribute only "little" to the error reduction can be identified and omitted from the approximation space [Bab60] and the dependence of the complexity on the dimension n may be weakened significantly.

This is related to the fact that the Fourier coefficients of certain multivariate functions exhibit decay properties that depend on the product of the frequencies in several coordinate directions. In contrast, the Fourier coefficients of functions from the standard (isotropic) Sobolev spaces behave like some power of the maximal frequency over all coordinate directions. In the first case, trigonometric approximation with cubes from the frequency space is inadequate, because it does not take into account these decay properties and leads to algorithms with unnecessary high cost. Instead, the approximation spaces should be adapted to the decay properties of the Fourier coefficients.

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This observation has been utilized in the development of hyperbolic cross spaces. There the frequencies of the trigonometric basis functions that are included into the approximation space form a hyperbolic cross in Fourier space. These hyperbolic cross spaces (also known as sparse grid spaces) have now a long tradition in approximation and interpolation theory [Sm63] and have been used not only in connection with trigonometric approximation/interpolation but also together with other multiscale tensor-product basis functions, like wavelets [Gri95, Sp98, DeV98, Bi99, Gri99, Gri00] or splines of higher order [Bu98]. Examples from the literature concerning approximation and interpolation include [Bab60, De82, Bas85, Gon89, Bas89, Hal92, Te93b], see also [Ka96]. Applications to the approximate solution of partial differential equations and integral equations can be found in [Ca76, Z91, Bu98, Bu99, Ho00] and [Gri98, Gri99], respectively. Generalizations of hyperbolic trigonometric approximation to more general domains in frequency space may be found in [Ga78, Di88] and [Gri98, Bu98, Gri00].
In this context, one of the basic problems of scientific computing is the efficient evaluation of integral expressions of the form

$$h(x) = Au(x) = \int_{T^n} k(x,y)u(y)dy \tag{1}$$

with smooth global kernel k. Standard algorithms that use regular full grids for the discretization of the kernel in (1) lead to a cost that is exponentially dependent on $2n$ (n dimension). This makes the cost of evaluating Au at some point x very expensive, even for small dimensions ($n = 2$ oder 3).
To reduce this cost, the idea is to exploit the smoothness of the kernel k and the function u (and hence also the smoothness of h). Then one can construct (modifications of) hyperbolic cross spaces such that the evaluation of integral expressions of the type (1) can be performed with much lower cost but the same accuracy as with the full grid approximation space, see [P89] and [F96] for special cases with isotropic Sobolev smoothness for the kernel and [F95] for dominating mixed smoothness for k und u.
In this paper we generalize the constructions of [P89, F95, F96] with respect to the considered smoothness assumptions and norms. The smoothness classes we consider are a mixture of Sobolev spaces of dominating mixed smoothness and isotropic smoothness, see Section 2 below. We construct approximation spaces of lower dimension than the standard full-grid spaces which preserve the approximation order of the standard full-grid spaces, provided that certain regularity assumptions are fulfilled. The idea is to represent the kernel k as well as u in a tensor-product multiscale basis and to translate the smoothness properties of the kernel into decay properties of the multiscale coefficients. Then these decay properties can be used to construct appropriate approximation spaces. Specifically we show under which circumstances one can break the curse of dimensionality and get rid of all asymptotic dependencies on n.
The resulting approximation spaces are in general not so easy to handle numerically. However, functions from these new approximation spaces may be represented as a sum of approximations on anisotropic full grids (of small dimension). This idea has been utilized in connection with hyperbolic cross spaces [Bas85, Bas89, Gon89] and can be easily extended to the more general case we consider here. In such a way, we

can circumvent the necessity of complicated data structures. Moreover, this approach enables the straightforward use of hierarchical methods such as multilevel schemes and fast Fourier transformations in much the same way as it has been done in [Bas89, Hal92] for hyperbolic cross spaces.
The remainder of the paper is organized as follows. Section 2 collects the necessary definitions on Sobolev spaces via Fourier series and defines the concept of smoothness we consider here. In Section 3 we define our trigonometric approximation spaces and we comment on the approximation properties of these spaces. Additionally we draw some conclusions for the complexity of resulting trigonometric approximation algorithms and discuss its implications for an optimal choice of approximation spaces. Section 4 shows how to construct blending schemes for our approximation spaces and in Section 5 we present some numerical examples. Finally, Section 6 contains some concluding remarks. Note that in the following we stick to trigonometric bases functions and the cube with periodic boundary conditions. However, our results can be applied to other $\mathcal{L}^2$-orthogonal tensor-product basis functions and even to biorthogonal tensor-product wavelets. Hence the results carry over to problems with other boundary conditions and more general domains provided that they allow tensor-product constructions.

2. Sobolev spaces and Fourier coefficients. We denote by $T^n = T \times T \times \cdots \times T$, $T := [0,1]$, the n-dimensional torus which is identified with the n-dimensional cube, where opposite faces are identified. We write $x \simeq y$ if there exist C_1, C_2 independent of any parameters x or y may depend on such that $C_1 \cdot y \leq x \leq C_2 \cdot y$. In the rest of the paper C denotes a generic constant which may depend on the smoothness assumptions and on the dimension n of the problem. Moreover, for $\mathbf{j}, \mathbf{k} \in \mathbb{N}^n$ we write $|\mathbf{j}|_\infty := \max_{1 \leq i \leq n}(|j_i|)$, $|\mathbf{j}|_1 := \sum_{i=1}^n |j_i|$, $|\mathbf{j}, \mathbf{k}|_\infty := \max(|\mathbf{j}|_\infty, |\mathbf{k}|_\infty)$ and $|\mathbf{j}, \mathbf{k}|_1 := |\mathbf{j}|_1 + |\mathbf{k}|_1$.
Let us define periodic Sobolev spaces in n dimensions via Fourier series, i.e. we classify functions via the decay of their Fourier coefficients. Obviously there are many possibilities to generalize the concept of Sobolev spaces in one dimension [A75] to higher dimensions. Two simple possibilities are the additive or multiplicative combination (or if you like an arithmetic or geometric averaging) of the frequencies in the different coordinate directions. We use the following definition that combines both possibilites. For $-\infty < t, l < \infty$, set

$$\mathcal{H}^{t,l}_{mix}(T^n) = \Bigg\{ u(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^n} c_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}} : \\ \|u\|_{\mathcal{H}^{t,l}_{mix}} = \left(\sum_{\mathbf{k} \in \mathbb{Z}^n} \prod_{i=1}^n (1 + |k_i|)^{2t} \cdot (1 + |\mathbf{k}|_\infty)^{2l} \cdot |c_{\mathbf{k}}|^2 \right)^{1/2} < \infty \Bigg\}. \quad (2)$$

The standard isotropic Sobolev spaces [A75] as well as the Sobolev spaces of dominating mixed smoothness [Schm87] are included here. They can be written as

$$\mathcal{H}^s(T^n) = \mathcal{H}^{0,s}_{mix}(T^n)$$

$$= \left\{ u(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} c_{\mathbf{k}} e^{-i\mathbf{kx}} : \|u\|_{\mathcal{H}^s} = \left(\sum_{\mathbf{k}\in\mathbb{Z}^n} (1+|\mathbf{k}|_\infty)^{2s} \cdot |c_{\mathbf{k}}|^2 \right)^{1/2} < \infty \right\} \quad (3)$$

and

$$\mathcal{H}^{t,0}_{mix}(T^n) = \left\{ u(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} c_{\mathbf{k}} e^{-i\mathbf{kx}} : \|u\|_{\mathcal{H}^{t,0}_{mix}} = \left(\sum_{\mathbf{k}\in\mathbb{Z}^n} \prod_{i=1}^n (1+|\mathbf{k}_i|)^{2t} \cdot |c_{\mathbf{k}}|^2 \right)^{1/2} < \infty \right\},$$

respectively. Hence, the parameter l from (2) governs the isotropic smoothness, whereas t governs the mixed smoothness. Thus, the spaces $\mathcal{H}^{t,l}_{mix}$ give us a quite flexible framework for the study of problems in Sobolev spaces.
Note that the spaces $\mathcal{H}^{t,l}_{mix}$ are a mixture of tensor-products [Wei80] of one-dimensional Sobolev spaces. For $l \geq 0, t \in \mathbb{R}$ it holds

$$\mathcal{H}^{t,l}_{mix}(T^n) = \mathcal{H}^{t\mathbf{1}+l\mathbf{e_1}}_{mix}(T^n) \cap \cdots \cap \mathcal{H}^{t\mathbf{1}+l\mathbf{e_n}}_{mix}(T^n), \quad (4)$$

where $\mathcal{H}^{\mathbf{k}}_{mix}(T^n) = \mathcal{H}^{k_1}(T) \otimes \ldots \otimes \mathcal{H}^{k_n}(T)$, $\mathbf{1} = (1,\ldots,1)$ and $\mathbf{e_i} = (0,\ldots,0,1,0,\ldots,0)$ is the i-th unit-vector in $\mathbb{R}^n$. This may easily be seen from the definition of the tensor-product via orthonormal systems and the intersection of spaces, compare [Gri95, Gri00]. See also [Ho00] for analogous constructions with more general boundary conditions. For negative isotropic smoothness index, i.e. $l \leq 0$, the space $\mathcal{H}^{t,l}_{mix}(T^n)$ may be equivalently defined as the dual of $\mathcal{H}^{-t,-l}_{mix}(T^n)$. Because of the relations $\prod_{i=1}^n (1+|k_i|)^{1/n} \leq 1+|\mathbf{k}|_\infty \leq \prod_{i=1}^n (1+|k_i|)$ and $1/\prod_{i=1}^n(1+|k_i|) \leq 1/(1+|\mathbf{k}|_\infty) \leq 1/\prod_{i=1}^n(1+|k_i|)^{1/n}$ the inclusions $\mathcal{H}^{t+l,0}_{mix} \subset \mathcal{H}^{t,l}_{mix} \subset \mathcal{H}^{t+l/n,0}_{mix}$ for $l \geq 0, t \in \mathbb{R}$, and $\mathcal{H}^{t+l/n,0}_{mix} \subset \mathcal{H}^{t,l}_{mix} \subset \mathcal{H}^{t+l,0}_{mix}$ for $l \leq 0, t \in \mathbb{R}$, hold. When the meaning is clear from the context, we will sometimes drop T^n and write for example $\mathcal{H}^{t,l}_{mix}$ instead of $\mathcal{H}^{t,l}_{mix}(T^n)$.
The important thing about (2) (at least in our context) is the characterization of $\mathcal{H}^{t,l}_{mix}$ via the decay properties of the Fourier coefficients $c_{\mathbf{k}}$. The definition (2) implies that for $u(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} c_{\mathbf{k}} e^{-i\mathbf{kx}} \in \mathcal{H}^{t,l}_{mix}$ one has $|c_{\mathbf{k}}| \leq C \cdot \left(\prod_{i=1}^n (1+|\mathbf{k}_i|)^t (1+|\mathbf{k}|_\infty)^l \right)^{-1}$ for almost all $c_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^n$.

3. Approximation spaces. Now we turn to the construction of approximation spaces for integral expressions of the form (1). We start with the approximation of functions in $\mathcal{H}^{t,l}_{mix}$.

3.1. Approximation in $\mathcal{H}^{t,l}_{mix}$. We first estimate the error between a function $u \in \mathcal{H}^{t,l}_{mix}$ and an approximation u_d. Let $I \subset \mathbb{Z}^n$, $u \in \mathcal{H}^{t,l}_{mix}$, $u(\mathbf{x}) = \sum_{\mathbf{k}} c_{\mathbf{k}} e^{-i\mathbf{kx}}$ and $u_d(\mathbf{x}) = \sum_{\mathbf{k}\in I} c_{\mathbf{k}} e^{-i\mathbf{kx}}$. Then it holds

$$\begin{aligned}
\|u-u_d\|^2_{\mathcal{H}^s} &= \|\sum_{\mathbf{k}\notin I} c_{\mathbf{k}} e^{-i\mathbf{k}\cdot}\|^2_{\mathcal{H}^s} \overset{(2)}{=} \sum_{\mathbf{k}\notin I} (1+|\mathbf{k}|_\infty)^{2s} |c_{\mathbf{k}}|^2 \\
&\leq \max_{\mathbf{k}\notin I} \left(\frac{(1+|\mathbf{k}|_\infty)^{2s}}{\prod_{i=1}^n (1+|k_i|)^{2t} (1+|\mathbf{k}|_\infty)^{2l}} \right) \cdot \sum_{\mathbf{k}\notin I} \prod_{i=1}^n (1+|k_i|)^{2t} (1+|\mathbf{k}|_\infty)^{2l} \cdot |c_{\mathbf{k}}|^2 \\
&\overset{(2)}{\leq} \max_{\mathbf{k}\notin I} \left(\frac{(1+|\mathbf{k}|_\infty)^{2(s-l)}}{\prod_{i=1}^n (1+|k_i|)^{2t}} \right) \cdot \|u\|^2_{\mathcal{H}^{t,l}_{mix}}. \qquad (5)
\end{aligned}$$

If we use in (5) for I the index set $\{\mathbf{j} \in \mathbb{Z}^n : |\mathbf{j}|_\infty \le d\}$ which corresponds to the standard approximation space with frequencies from the cube $[-d, d]^n$, we obtain

$$\|u - u_d\|_{\mathcal{H}^s} \le (1 + d)^{s-l-t} \cdot \|u\|_{\mathcal{H}^{t,l}_{mix}}. \tag{6}$$

A close look on (5) reveals that we can discard indices from the full grid without destroying the order of approximation in (6). This motivates the following definition of index sets and approximation spaces.
Let $d \in \mathbb{N}$, $T \in (-\infty, 1]$. Let the index sets FI_d^T and the spaces FV_d^T be given by

$$FI_d^T \;:=\; \left\{\mathbf{k} \in \mathbb{Z}^n : \prod_{i=1}^{n}(1 + |k_i|) \cdot (1 + |\mathbf{k}|_\infty)^{-T} \le (1 + d)^{1-T}\right\}, \tag{7}$$

$$FV_d^T \;:=\; \operatorname{span}\{e^{-i\mathbf{k}\cdot} : \mathbf{k} \in FI_d^T\} = \{f(\mathbf{x}) = \sum_{\mathbf{k}\in FI_d^T} c_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}}\}. \tag{8}$$

As a natural extension to the case $T = -\infty$ we define

$$FI_d^{-\infty} := \{\mathbf{k} \in \mathbb{Z}^n : |\mathbf{k}|_\infty \le d\}$$

and similarly $FV_d^{-\infty}$.
The new parameter T in (7) allows us to decouple the definition of the index sets from the smoothness parameters and to investigate more closely into the relation between smoothness assumptions, the choice of approximation space and the order of approximation. The two parameters d and T govern the number of elements in the approximation space as well as the choice of frequencies that are selected from the whole spectrum. The upper bound $(1 + d)^{1-T}$ limits the maximal frequency (in every direction) in our approximation spaces to d. Figure 1 shows the frequencies of those exponentials that are included into the approximation space FV_d^T for various parameters T in the two-dimensional case. The well known hyperbolic cross [Sm63] appears as a special case for $T = 0$. A positive value of T leads to a smaller approximation space compared to the hyperbolic cross space, whereas a negative value of T leads to a larger approximation space. The standard approximation space connected with the regular full grid is obtained with $T = -\infty$. Hence the spaces FV_d^T from (8) form a scale of nested approximation spaces

$$FV_d^{T_1} \subset FV_d^{T_2} \subset FV_d^{-\infty}, \text{ for } -\infty < T_2 < T_1 \le 1,$$

including the hyperbolic cross spaces as well as the standard full grid approximation spaces.
The following Theorem summarizes the approximation properties of the spaces FV_d^T.

THEOREM 1. *Let* $\epsilon \in \mathbb{R}^+$, $s < l + t, t \ge 0$, $u \in \mathcal{H}^{t,l}_{mix}$, $u(\mathbf{x}) = \sum_{\mathbf{k}} c_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}}$ *and* $u_d^T(\mathbf{x}) = \sum_{\mathbf{k}\in FI_d^T} c_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}}$ *its approximation in* FV_d^T. *Then it holds*

$$\|u - u_d^T\|_{\mathcal{H}^s} \;\le\; \begin{cases} O\left((1 + d)^{s-l-t+(Tt-s+l)\frac{n-1}{n-T}}\right) \cdot \|u\|_{\mathcal{H}^{t,l}_{mix}} & \text{for } T \ge \frac{s-l}{t}, \\ O\left((1 + d)^{s-l-t}\right) \cdot \|u\|_{\mathcal{H}^{t,l}_{mix}} & \text{for } T \le \frac{s-l}{t}. \end{cases} \tag{9}$$

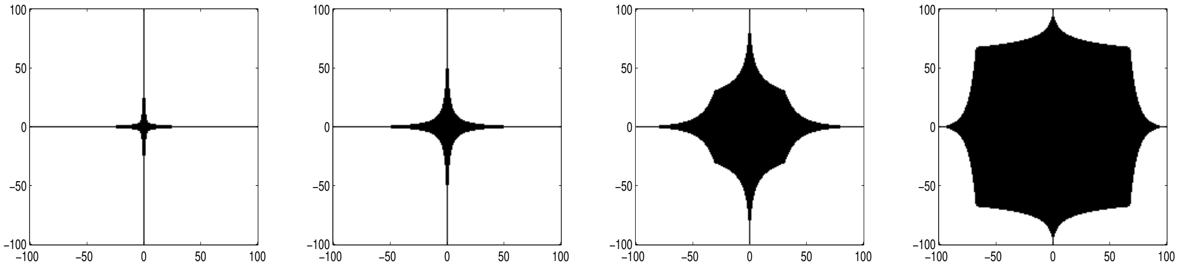

FIG. 1. *Index sets FI_{100}^T for $T = 0.5, 0, -2, -10$ (left to right) in the two-dimensional case; the hyperbolic cross corresponds to $T = 0$*

The number of elements $\mathcal{N}$ needed to obtain an approximation u_d with $\|u - u_d\|_{\mathcal{H}^s} \le \epsilon$ is bounded by

$$\mathcal{N} \le \begin{cases} O\left(\epsilon^{\frac{1}{s-l-t}}\right) & \text{for } s > l, t > 0, \\ O\left(\epsilon^{\frac{1}{s-l-t}} \cdot \ln(\epsilon^{-1})^{n-1}\right) & \text{for } s = l, t > 0, \\ O\left(\epsilon^{\frac{1}{\frac{s-l}{n}-t}}\right) & \text{for } s < l, t > 0, \\ O\left(\epsilon^{\frac{n}{s-l}}\right) & \text{for } t = 0. \end{cases} \tag{10}$$

Proof: Inequality (9) follows from (7) together with (5). We note that the estimates

$$\inf_{v \in V_J^T} \|u - v\|_{\mathcal{H}^s} \le \begin{cases} O\left(\mathcal{N}^{s-l-t}\right) \cdot \|u\|_{\mathcal{H}^{t,l}_{mix}} & \text{for } 0 < T, T < \frac{s-l}{t}, \\ O\left(\mathcal{N}^{s-l-t+(Tt-s+l)\frac{n-1}{n-T}}\right) \cdot \|u\|_{\mathcal{H}^{t,l}_{mix}} & \text{for } 0 < T, T \ge \frac{s-l}{t}, \\ O\left(\left(\frac{\mathcal{N}}{\ln(\mathcal{N})^{n-1}}\right)^{s-l-t}\right) \cdot \|u\|_{\mathcal{H}^{t,l}_{mix}} & \text{for } T = 0, T < \frac{s-l}{t}, \\ O\left(\left(\frac{\mathcal{N}}{\ln(\mathcal{N})^{n-1}}\right)^{\frac{s-l}{n}-t}\right) \cdot \|u\|_{\mathcal{H}^{t,l}_{mix}} & \text{for } T = 0, T \ge \frac{s-l}{t}, \\ O\left(\mathcal{N}^{\frac{s-l-t}{n}\cdot\frac{T-n}{T-1}}\right) \cdot \|u\|_{\mathcal{H}^{t,l}_{mix}} & \text{for } T < 0, T < \frac{s-l}{t}, \\ O\left(\mathcal{N}^{\frac{s-l}{n}-t}\right) \cdot \|u\|_{\mathcal{H}^{t,l}_{mix}} & \text{for } T < 0, T \ge \frac{s-l}{t}, \\ O\left(\mathcal{N}^{\frac{s-l-t}{n}}\right) \cdot \|u\|_{\mathcal{H}^{t,l}_{mix}} & \text{for } T = -\infty, \end{cases} \tag{11}$$

are a direct consequence of (9) together with the following estimate of the number of elements in FV_d^T,

$$dim\left(FV_d^T\right) \le \begin{cases} O(1+d) & \text{for } 0 < T \le 1, \\ O\left((1+d)\cdot\ln(1+d)^{n-1}\right) & \text{for } T = 0, \\ O\left((1+d)^{\frac{T-1}{T/n-1}}\right) & \text{for } T < 0, \\ O\left((1+d)^n\right) & \text{for } T = -\infty. \end{cases} \tag{12}$$

For a proof of (12) see [Di83, Gri00]. Now we have to assume that the right hand sides in (11) are smaller than ϵ and to solve for $\mathcal{N}$. The choice $T = \frac{s-l}{t}$ then shows (10).

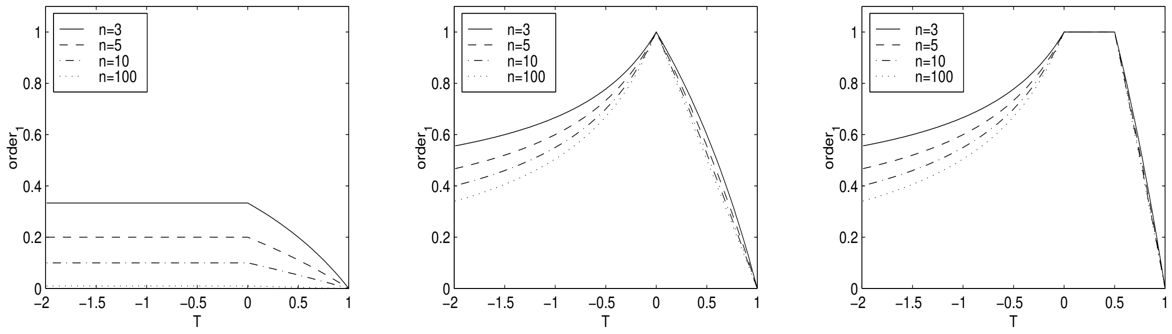

FIG. 2. order$_1$ *for* $s=1$ *and the spaces* $\mathcal{H}^{0,2}_{mix}$, $\mathcal{H}^{1,1}_{mix}$ *and* $\mathcal{H}^{2,0}_{mix}$ *(left to right);*

$\square$

Theorem 1 shows that the optimal order of approximation is kept when changing from the full grid approximation space $FV_d^{-\infty}$ to an approximation space FV_d^T with $T \leq (s-l)/t$. However, the use of approximation spaces FV_d^T with $T > (s-l)/t$ leads to a deterioration of the optimal order of convergence. Moreover we obtain from (10) that the number of elements necessary to obtain an ϵ-approximation of a function from $\mathcal{H}^{t,l}_{mix}$ is asymptotically independent of the dimension n if $s < l, t > 0$ hold.

The relevant sizes in the estimates in (11) are the exponents of $\mathcal{N}$. They are a measure for the approximation power of our approximation spaces. If we disregard eventual logarithmic factors, we are led to a nice possibility to visualize the implications of Theorem 1 by plotting the negative $\mathcal{N}$-exponents in (11). We denote these by order$_1$ and plot them versus T for different n, different smoothness classes $\mathcal{H}^{t,l}_{mix}$ and different error-norms $\|.\|_{\mathcal{H}^s}$. Figures 2 and 3 show some examples.

For the standard Sobolev space $u \in \mathcal{H}^{0,2}_{mix}$ (i.e. $t=0, l=2$). and the approximation spaces with $T \leq 0$ the resulting order is independent of T but dependent on the dimension n. This reflects the so called 'curse of dimensionality' which makes problems in isotropic Sobolev spaces (3) intractable for higher dimensions n, see Figures 2 and 3 (left). For $T > 0$ the order deteriorates with larger T.

If we restrict the class of functions, now to the smoothness class $\mathcal{H}^{1,1}_{mix}$ (i.e. $t=1, l=1$) we obtain the results of Figures 2 and 3 (middle). For example Figure 2 (middle) shows that for $s=1$ (i.e. the error is measured in the $\mathcal{H}^1$-norm) the order is dependent on the dimension n for all T. But for $T=0$ the dependence is only logarithmically.

For the smoothness class $\mathcal{H}^{2,0}_{mix}$ (i.e. $t=2, l=0$) the situation is different again. For example for $T < 0$ and $s=1$, the complexities are again dependent on n, but in the range $0 < T \leq 1/2$ the order is independent of T and n, see Figure 2 (right). For $T > 1/2$ the order deteriorates.

Similarly Figure 3 shows the corresponding graphs for the case $s = -1/2$. Again, the dependence on the dimension n is reduced with higher smoothness. However, there remains a slight n-dependence also for the case of higher (mixed) smoothness, which is in contrast to the case of positive s.

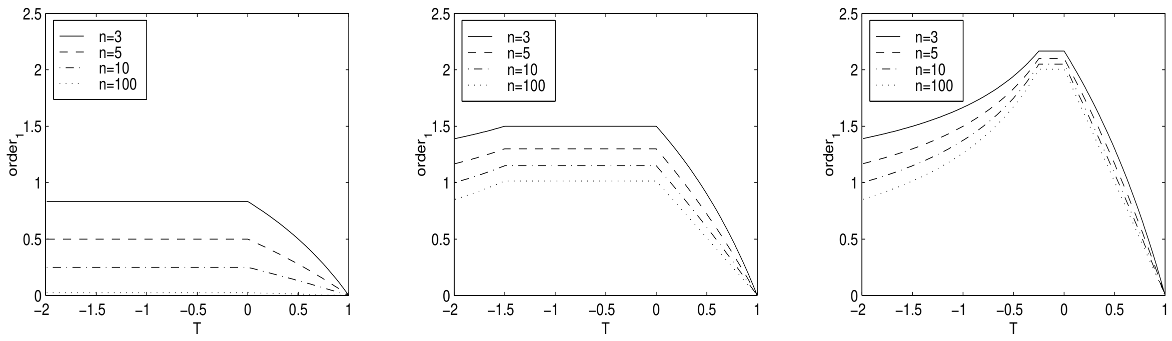

FIG. 3. order$_1$ *for* $s = -\frac{1}{2}$ *and the spaces* $\mathcal{H}^{0,2}_{mix}$, $\mathcal{H}^{1,1}_{mix}$ *and* $\mathcal{H}^{2,0}_{mix}$ *(left to right);*

3.2. Kernel approximation. Now we turn to the approximation of the integral operator A from (1), i.e. we look for approximation spaces for k, u and h in (1). We write $\psi_{\mathbf{j}}(x) = \left(\frac{1}{2\pi}\right)^{n/2} e^{-i\mathbf{j}\mathbf{x}}$, $\mathbf{j} \in \mathbb{Z}^n$. Let $k(x,y) = \sum_{\mathbf{jk}} a_{\mathbf{jk}} \psi_{\mathbf{j}}(x)\psi_{\mathbf{k}}(y)$, $u(x) = \sum_{\mathbf{k}} b_{\mathbf{k}} \psi_{\mathbf{k}}(x)$ and let there be given an approximation of the kernel of the form

$$k_d(x,y) = \sum_{(\mathbf{j},\mathbf{k})\in I} a_{\mathbf{jk}} \psi_{\mathbf{j}}(x)\psi_{\mathbf{k}}(y)$$

with some index set $I \subset \mathbb{Z}^{2n}$. We define the operator A_d by

$$A_d u(x) = \int_{T^n} k_d(x,y) u(y) dy. \tag{13}$$

The following Theorem estimates the error $\|(A - A_d)u\|_{\mathcal{H}^s}$ between the integral operator A and its approximation A_d.

THEOREM 2. *Let* $(A - A_d)u \in \mathcal{H}^s$, $s \in \mathbb{R}$. *Then*

$$\|(A - A_d)u\|^2_{\mathcal{H}^s} \simeq \sum_{\mathbf{j}} (1 + |\mathbf{j}|_\infty)^{2s} \left(\sum_{\mathbf{k}:(\mathbf{j},\mathbf{k})\notin I} a_{\mathbf{jk}} b_{\mathbf{k}} \right)^2. \tag{14}$$

Let moreover $u \in \mathcal{H}^{p,q}_{mix}(T^n)$ *and* $k \in \mathcal{H}^{t,l}_{mix}(T^{2n})$, *with* $p, q, t, l \in \mathbb{R}$, *then it holds*

$$\|(A - A_d)u\|_{\mathcal{H}^s} \le C \cdot \left(\max_I\right) \cdot \|k\|_{\mathcal{H}^{t,l}_{mix}} \cdot \|u\|_{\mathcal{H}^{p,q}_{mix}}, \tag{15}$$

where

$$\max_I := \max_{(\mathbf{j},\mathbf{k})\notin I} \left(\frac{(1 + |\mathbf{j}|_\infty)^s}{(1 + |\mathbf{j},\mathbf{k}|_\infty)^l (1 + |\mathbf{k}|_\infty)^q \cdot \prod_{i=1}^n (1 + |j_i|)^t (1 + |k_i|)^{t+p}} \right).$$

Proof: Due to the $\mathcal{L}^2$-orthonormality and the stability of the basis in $\mathcal{H}^s$ we have

$$\|(A - A_d)u\|^2_{\mathcal{H}^s} = \| \int \left(\sum_{(\mathbf{j},\mathbf{k})\notin I} a_{\mathbf{jk}} \psi_{\mathbf{j}}(x)\psi_{\mathbf{k}}(y) \right) \cdot \left(\sum_{\mathbf{l}} b_{\mathbf{l}} \psi_{\mathbf{l}}(y) \right) dy \|^2_{\mathcal{H}^s}$$

$$
\begin{aligned}
&= \|\sum_{(\mathbf{j},\mathbf{k})\notin I} a_{\mathbf{jk}} b_{\mathbf{k}} \psi_{\mathbf{j}}(x)\|^2_{\mathcal{H}^s} \\
&= \|\sum_{\mathbf{j}} \left(\sum_{\mathbf{k}:(\mathbf{j},\mathbf{k})\notin I} a_{\mathbf{jk}} b_{\mathbf{k}} \right) \psi_{\mathbf{j}}(x)\|^2_{\mathcal{H}^s} \\
&\overset{(3)}{\simeq} \sum_{\mathbf{j}} (1+|\mathbf{j}|_\infty)^{2s} \left(\sum_{\mathbf{k}:(\mathbf{j},\mathbf{k})\notin I} a_{\mathbf{jk}} b_{\mathbf{k}} \right)^2 . \qquad (16)
\end{aligned}
$$

Under the assumptions $u \in \mathcal{H}^{p,q}_{mix}(T^n)$ and $k \in \mathcal{H}^{t,l}_{mix}(T^{2n})$ we obtain

$$
\begin{aligned}
\|(A-A_d)u\|^2_{\mathcal{H}^s} &\simeq \sum_{\mathbf{j}} (1+|\mathbf{j}|_\infty)^{2s} \left(\sum_{\mathbf{k}:(\mathbf{j},\mathbf{k})\notin I} a_{\mathbf{jk}} b_{\mathbf{k}} \right)^2 \\
&\le \left(\max_I\right)^2 \cdot \sum_{\mathbf{j}} \left(\sum_{\mathbf{k}:(\mathbf{j},\mathbf{k})\notin I} (1+|\mathbf{j},\mathbf{k}|_\infty)^l \left(\prod_{i=1}^n (1+|j_i|)^t (1+|k_i|)^t \right) a_{\mathbf{jk}} \cdot \right. \\
&\qquad \left. \cdot (1+|\mathbf{k}|_\infty)^q \prod_{i=1}^n (1+|k_i|)^p b_{\mathbf{k}} \right)^2 \\
&\le \left(\max_I\right)^2 \cdot \sum_{\mathbf{j}} \left(\sum_{\mathbf{k}} (1+|\mathbf{j},\mathbf{k}|_\infty)^{2l} \left(\prod_{i=1}^n (1+|j_i|)^{2t} (1+|k_i|)^{2t} \right) a^2_{\mathbf{jk}} \right) \\
&\qquad \cdot \left(\sum_{\mathbf{k}} (1+|\mathbf{k}|_\infty)^{2q} \prod_{i=1}^n (1+|k_i|)^{2p} b^2_{\mathbf{k}} \right) \\
&\overset{(2)}{\simeq} \left(\max_I\right)^2 \cdot \|k\|^2_{\mathcal{H}^{t,l}_{mix}} \cdot \|u\|^2_{\mathcal{H}^{p,q}_{mix}} .
\end{aligned}
$$

This shows (15).
□

In the rest of this section we assume $l, t, p, q \ge 0$ and $l + t \ge s \ge -(p+q)$. Then, if we use in (15) for I the index set $\{(\mathbf{j},\mathbf{k}) \in \mathbb{Z}^{2n} : |\mathbf{j},\mathbf{k}|_\infty \le d\}$ which corresponds to the standard approximation space with frequencies from the cube $[-d,d]^{2n}$, we obtain

$$
\|(A-A_d)u\|_{\mathcal{H}^s} \le C \cdot (1+d)^{s-l-t} \cdot \|k\|_{\mathcal{H}^{t,l}_{mix}} \cdot \|u\|_{\mathcal{H}^{p,q}_{mix}} . \qquad (17)
$$

Note that this index set has $O((1+d)^{2n})$ elements. Hence the integral evaluation would require $O((1+d)^{2n})$ operations. However, a close look on (15) and (17) reveals that we can discard indices from the full grid index set without destroying the order of approximation. We see from (15) that the smallest index set that preserves the approximation properties of the standard approximation space is given by

$$
\begin{aligned}
FKI_d(s,t,l,p,q) &= \{(\mathbf{j},\mathbf{k}) \in \mathbb{Z}^{2n} : (1+|\mathbf{j}|_\infty)^{-s} (1+|\mathbf{k}|_\infty)^q (1+|\mathbf{j},\mathbf{k}|_\infty)^l \cdot \\
&\qquad \cdot \left(\prod_{i=1}^n (1+|j_i|)^t (1+|k_i|)^{t+p} \right) \le (1+d)^{l+t-s} \}. \qquad (18)
\end{aligned}
$$

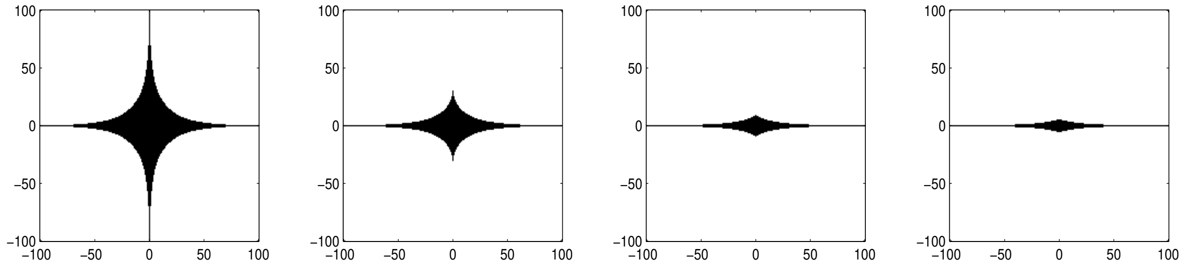

FIG. 4. *Example 2: Index sets* $FKI_{100}(s,r,0,r,0)$ *for* $s/r = -1, -0.5, 0, 0.2$ *(left to right) one-dimensional case;*

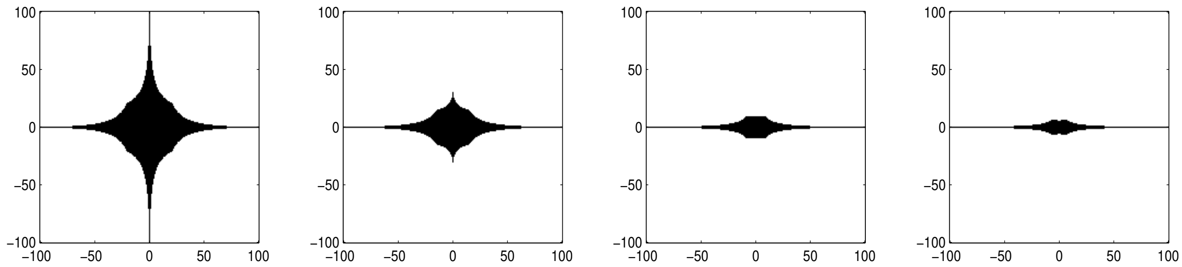

FIG. 5. *Example 1: Index set* $FKI_{100}(s,0,r,0,r)$ *for* $s/r = -1, -0.5, 0, 0.2$ *(left to right) one-dimensional case;*

The corresponding trigonometric approximation space is $FKV_d(s,t,l,p,q) := \text{span}\{e^{-i\mathbf{jx}} \cdot e^{-i\mathbf{ky}} : (\mathbf{j},\mathbf{k}) \in FKI_d(s,t,l,p,q)\}$. The parameter d and the smoothness parameters s, l, t, p and q govern the number of elements in this approximation space as well as the choice of frequencies that are selected from the whole spectrum. The upper bound $(1+d)^{l+t-s}$ limits the maximal frequency (in every direction) in our approximation spaces to d.

Figures 4 and 5 show the index sets $FKI_d(s,r,0,r,0)$ and $FKI_d(s,0,r,0,r)$ in the one dimensional case for various choices of s and r. Note that in these cases, the resulting index sets depend only on the quotient s/r.

A special case is the index set $FKI_d(0,t,0,0,0) = \{(\mathbf{j},\mathbf{k}) : \prod_{i=1}^{n}(1+j_i)(1+k_i) \leq d\}$ for $t > 0$, corresponding to the (standard) hyperbolic cross. It is optimal for the approximation of the kernel if the error $\|(A - A_d)u\|$ is measured in the $\mathcal{L}^2$-norm and if we have $k \in \mathcal{H}^{t,0}_{mix}(T^{2n})$ and $u \in \mathcal{L}^2(T^n)$. Another special case is the index set $FKI_d(0,0,l,0,0) = \{(\mathbf{j},\mathbf{k}) : (1 + |\mathbf{j},\mathbf{k}|_\infty) \leq d\}$ for $l > 0$, corresponding to the full grid approximation space. It is optimal if the error is measured in the $\mathcal{L}^2$-norm and if we have $k \in \mathcal{H}^l(T^{2n})$ and $u \in \mathcal{L}^2(T^n)$.

The index set FKI_d from (18) can now be used to compute an approximation $h_d = \sum_{\mathbf{j}} h_{\mathbf{j}}\psi_{\mathbf{j}}(x)$ of h via the expression

$$h_{\mathbf{j}} = \sum_{\mathbf{k}:(\mathbf{j},\mathbf{k})\in FKI_d} a_{\mathbf{jk}} b_{\mathbf{k}}. \tag{19}$$

Note that the index set $FKI_d(s,t,l,p,q)$ for the kernel k implies index sets for the approximation of u and h in (1). This may be seen directly from (19). The corresponding index sets are

$$\{\mathbf{k} \in \mathbb{Z}^n : (\mathbf{0}, \mathbf{k}) \in FKI_d(s,t,l,p,q)\} \text{ for } u \text{ and} \tag{20}$$

$$FI_d^{\frac{s-l}{t}} = \{\mathbf{j} \in \mathbb{Z}^n : (\mathbf{j}, \mathbf{0}) \in FKI_d(s,t,l,p,q)\} \text{ for } h. \tag{21}$$

The following Lemma summarizes the approximation properties of our new approximation spaces.

LEMMA 1. *Let $s \in \mathbb{R}, t, l, p, q \in \mathbb{R}_0^+$, $t+l \geq s \geq -(p+q)$, $k \in \mathcal{H}_{mix}^{t,l}(T^{2n})$, $u \in \mathcal{H}_{mix}^{p,q}(T^n)$ and $\epsilon \in \mathbb{R}^+$. Let A, u, h, A_d as above, where we take A_d as in (13) with*

$$k_d(x,y) = \sum_{(\mathbf{j},\mathbf{k}) \in FKI_d(s,t,l,p,q)} a_{\mathbf{jk}} \psi_{\mathbf{j}}(x) \psi_{\mathbf{k}}(y).$$

Moreover let $h_d(x) = \sum_{\mathbf{j} \in FI_d^{\frac{s-l}{t}}} h_{\mathbf{j}} \psi_{\mathbf{j}}(x)$ an approximation of h where the coefficients $h_{\mathbf{j}}$ are defined via (19). Then it holds

$$\|h - h_d\|_{\mathcal{H}^s} \leq C \cdot (1+d)^{s-l-t} \cdot \|k\|_{\mathcal{H}_{mix}^{t,l}} \cdot \|u\|_{\mathcal{H}_{mix}^{p,q}}. \tag{22}$$

In the case $t > 0$, i.e. for a kernel with dominating mixed smoothness, the number of elements $\mathcal{N}$ needed to obtain an approximation h_d with $\|h - h_d\|_{\mathcal{H}^s} \leq \epsilon$ is bounded by

$$\mathcal{N} \leq \begin{cases} O(\epsilon^{\frac{1}{s-l-t}}) & \text{for } s-l > 0, \\ O(\epsilon^{\frac{1}{s-l-t}} \ln(\epsilon^{-1})^{n-1}) & \text{for } s-l = 0, \\ O\left(\epsilon^{\frac{1}{\frac{s-l}{n}-t}}\right) & \text{for } s-l < 0, \end{cases} \tag{23}$$

compare (10). In the case of isotropic smoothness, i.e. $t = p = 0$, the number of elements $\mathcal{N}$ needed to obtain an approximation h_d with $\|h - h_d\|_{\mathcal{H}^s} \leq \epsilon$ is bounded by

$$\mathcal{N} \leq \begin{cases} O(\epsilon^{\frac{n}{s-l}}) & \text{for } s-l+q > 0, \\ O(\epsilon^{\frac{n}{s-l}} \ln(\epsilon^{-1})) & \text{for } s-l+q = 0, \\ O\left(\epsilon^{\frac{2n}{s-l-q}}\right) & \text{for } s-l+q < 0. \end{cases} \tag{24}$$

Proof: The inequality (22) follows directly from (15) and the definition of FKI_d in (18). To proof (23) and (24) compute the dimension of the index set $FKI_d(s,t,l,p,q)$ (apply Theorem 6 from [Di83]), plug these estimates into (22), assume that the right hand sides of the resulting estimates are smaller than ϵ and solve for $\mathcal{N}$.
□

Inequality (22) repeats the fact that the index set $FKI_d(s,t,l,p,q)$ results in the same error as the standard full grid approximation space, although the dimension of $FKI_d(s,t,l,p,q)$ is (in most cases) much smaller.
Interestingly, even in the case of kernel with isotropic smoothness, sparse grid constructions are more suitable for the approximation than the full grid. This has been discovered in [P89] for the special case $k \in \mathcal{H}^r(T^{2n})$, $u \in \mathcal{H}^r(T^n)$ and $s = 0$, see also [F96].

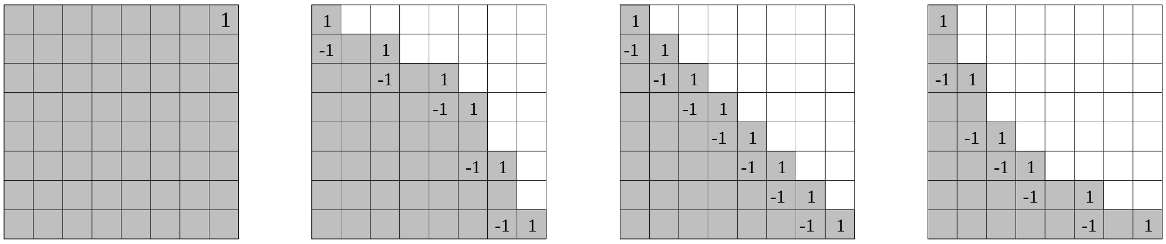

FIG. 6. *Index sets $BI_8^{-\infty}, BI_8^{-1}, BI_8^{0}$ and $BI_8^{0.5}$ in grey shading (left to right) and coefficients $r_{\mathbf{j}}$ from (29), 2D case. Only the coefficients $r_{\mathbf{j}} \neq 0$ are shown.*

4. Blending-schemes. In this section we construct an algorithm for the evaluation of integral expressions of the type (1) that uses the index sets FKI_d from (18). The essential idea is to construct our approximation spaces as sums and differences of simpler spaces via blending schemes. Such constructions are of special interest, as our approximation spaces are in general not so easy to handle numerically as standard spaces connected with full grid point sets. Moreover, this approach enables the straightforward use of the fast Fourier transform and circumvents the necessity of complicated data structures. See [Bas85, Bas89, Gon89] for similar constructions for the space FV_d^0. For implementational reasons it is convenient to use dyadic refinement and dyadic hyperbolic crosses. Let

$$\begin{aligned} G_J(s,t,l,p,q) \;&:=\; \{(\mathbf{j},\mathbf{k}) \in \mathbb{N}^{2n} : \\ &\quad -s|\mathbf{j}|_\infty + l|\mathbf{j},\mathbf{k}|_\infty + t|\mathbf{j},\mathbf{k}|_1 + q|\mathbf{k}|_\infty + p|\mathbf{k}|_1 \le (l+t-s)J\}. \end{aligned} \tag{25}$$

Then, instead of the index sets from (18) we use

$$\begin{aligned} I_J(s,t,l,p,q) \;&:=\; \{(\mathbf{j},\mathbf{k}) \in \mathbb{N}^{2n} : \; \exists(\mathbf{v},\mathbf{w}) \in G_J(s,t,l,p,q) \\ &\qquad \text{such that } j_i \le 2^{v_i}, k_i \le 2^{w_i}, 1 \le i \le n\} \end{aligned} \tag{26}$$

and instead of FI_d^T from (7) we use

$$BI_J^{\frac{s-l}{t}} \;:=\; \{\mathbf{j} \in \mathbb{N}^n : (\mathbf{j},\mathbf{0}) \in I_J(s,t,l,p,q)\}, \tag{27}$$

where J denotes the level of refinement. The statements from section 3 on the dimension of our spaces and their approximation properties carry over directly (just substitute $(1+d)$ by 2^J in Lemma 1 and Theorem 1). Figure 6 shows some examples for the index sets BI_J^T from (27) in the two-dimensional case.
Our algorithms are based on the following Blending-Theorem.

THEOREM 3. *Let $I \subset \mathbb{N}^N, N > 1$ have the following property: If $\mathbf{j} \in I$, then every $\mathbf{k}$ with $\mathbf{k} \le \mathbf{j}$ is in I.*
Let $W_{\mathbf{j}} = span\{\psi_{\mathbf{k}} : \mathbf{k} \le \mathbf{j}\}$, $V = \sum_{\mathbf{j}\in I} W_{\mathbf{j}}$ and $u = \sum_{\mathbf{j}\in I} w_{\mathbf{j}},\ w_{\mathbf{j}} \in W_{\mathbf{j}}$. Let $V_{\mathbf{j}} = \sum_{\mathbf{k}\le\mathbf{j}} W_{\mathbf{k}}$ be the anisotropic full grid space and let

$$u_{\mathbf{j}} = \sum_{\mathbf{k}\le\mathbf{j}} w_{\mathbf{k}} \text{ for } \mathbf{j} \in I. \tag{28}$$

Then we have

$$u = \sum_{\mathbf{j}\in I: r_{\mathbf{j}}\neq 0} r_{\mathbf{j}} u_{\mathbf{j}} \quad \textit{with} \quad r_{\mathbf{j}} = \sum_{k=0}^{N-1} (-1)^k \left(\sum_{\substack{\{\mathbf{l}:\mathbf{l}=\mathbf{j}+\sum_{i\in M}\mathbf{e}_i\in I,\\ M\subseteq\{1,\ldots,N\}, |M|=k\}}} 1 \right). \tag{29}$$

That is, every element u from V can be written as a weighted sum of approximations $u_{\mathbf{j}}$ from anisotropic full grid approximation spaces $V_{\mathbf{j}}$.

To proof (29), note that every element $w_{\mathbf{j}}$ from $W_{\mathbf{j}}$ may be written as sum and difference of elements $u_{\mathbf{k}} = \sum_{\mathbf{k}\leq\mathbf{j}} w_{\mathbf{j}}$ from the spaces $V_{\mathbf{k}}, \mathbf{k}\leq\mathbf{j}$,

$$w_{\mathbf{j}} \;\; = \;\; \sum_{k=0}^{N} (-1)^k \sum_{\substack{\{\mathbf{l}:\mathbf{l}=\mathbf{j}-\sum_{i\in M}\mathbf{e}_i\in I,\\ M\subseteq\{1,\ldots,N\}, |M|=k\}}} u_{\mathbf{l}}.$$

Now we plug this into $u = \sum_{\mathbf{j}\in I} w_{\mathbf{j}}$, collect equal terms in the sum and exploit the fact that $r_{\mathbf{j}} = 0$ if $\mathbf{j}+\mathbf{1}\in I$.
Now we apply Theorem 3 for the interpolation of smooth functions and the evaluation of integral expressions.

Example 1: Interpolation
For trigonometric interpolation and also for integration purposes one needs sampling/interpolation points in physical space. Our approximation spaces imply certain grid sequences that can be used for these purposes. For $\mathbf{j} = (j_1,\ldots,j_N), N\in\mathbb{N}$ we define the anisotropic full grid

$$S_{\mathbf{j}} := \left\{ (a_1 2^{-j_1},\ldots,a_n 2^{-j_N}), a_i = 0,\ldots,2^{j_i}-1, 1\leq i\leq N \right\}$$

The grid sequence associated with an index set $I\subset\mathbb{N}^N$ is

$$S_I := \cup_{\mathbf{j}\in I} S_{\mathbf{j}}. \tag{30}$$

Note that the number of grid points in S_I is given by $dim(I)$. For example for the index set BI_J^T from (27) the corresponding grid point set is given by $S_{BI_J^T} := \cup_{\mathbf{j}\in BI_J^T} S_{\mathbf{j}}$. These grid point sets are nested and include the full grid ($T = -\infty$) and the grids connected with the hyperbolic cross spaces ($T = 0$) as special cases. Figure 7 shows some examples of $S_{BI_J^T}$ in the two-dimensional case.
The following algorithm may be used for the interpolation of a (sufficiently smooth) function f at the grid points S_I from (30). The algorithm is an extension of the algorithm described in [Bas89, Hal92] for the interpolation on standard hyperbolic crosses FI_d^0.

Algorithm 1. *(Interpolation)*

1. *Sample function values of f at the grid points of those anisotropic grids $S_{\mathbf{j}}$ that have $\mathbf{j}\in I, r_{\mathbf{j}}\neq 0$.*

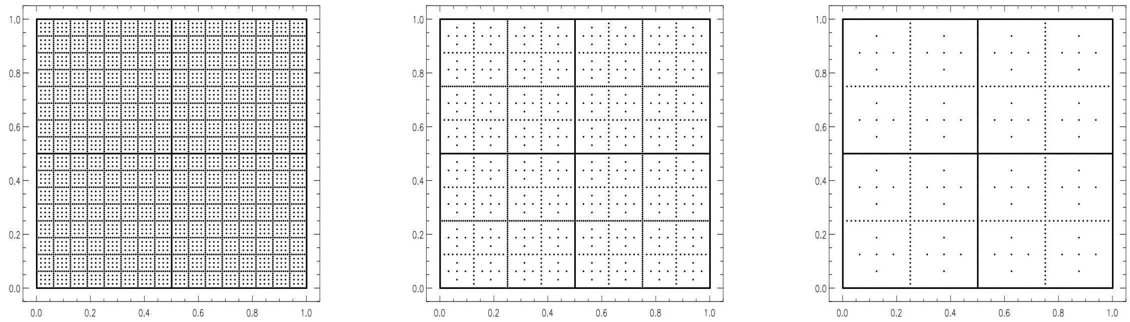

FIG. 7. *Grid points* $S_{BI_8^{-2}}, S_{BI_8^{0}}, S_{BI_8^{0.5}}$ *in the two-dimensional case*

2. *For each such grid Fourier transform the obtained grid values. This results in the coefficients* $c_{\mathbf{m}}$ *of the functions* $u_{\mathbf{j}} = \sum_{\mathbf{l} \leq \mathbf{j}} c_{\mathbf{l}} \psi_{\mathbf{l}} \in V_{\mathbf{j}}$ *from (28).*
3. *Combine the* $u_{\mathbf{j}}$ *according to (29).*

Step 2 may be performed by the fast Fourier transform [Co65].
A short consideration shows that the number of grid points $\sum_{\mathbf{j} \in I} |S_{\mathbf{j}}|$ that are used in the blending scheme is of the same order as $dim(I)$. Hence, the cost of algorithms using these blending schemes is bounded by the sum of the cost of the multiscale transformations on the anisotropic grids $S_{\mathbf{j}}, \mathbf{j} \in I$.

Example 2: Integral evaluation
The following algorithm may be used for the approximate evaluation of integrals of type (1) with (sufficiently) smooth kernel k and (sufficiently) smooth function u. We make use of Algorithm 1 to obtain interpolants of k and u. Now we use blending schemes for the approximation of k, u and h.
ALGORITHM 2. *(Integral evaluation)*

1. *Compute the blending representations of the interpolants of* k *and* u *according to Algorithm 1. For* k *use the index set* $I_J(s,t,l,p,q)$ *from (26) and for* u *use the index set* $\{\mathbf{l} : (\mathbf{0}, \mathbf{l}) \in I_J(s,t,l,p,q)\}$*, compare (20). This results in the coefficients* $a_{\mathbf{jk}}$ *for the approximation of* k *and* $b_{\mathbf{j}}$ *for the approximation of* u*.*
2. *Compute the coefficients* $h_{\mathbf{j}}, \mathbf{j} \in BI_J^{\frac{s-l}{t}}$ *(compare (27)) of the approximation of* h *according to* $h_{\mathbf{j}} = \sum_{\mathbf{k}:(\mathbf{j},\mathbf{k}) \in I_J} a_{\mathbf{jk}} b_{\mathbf{k}}$*, compare (19).*

The number of grid points that are used in this algorithm is of the order $O(|I_J(s,t,l,p,q)|)$. Note that Algorithms 1 and 2 are naturally dimension recursive and are as such directly usable in arbitrary dimensions.

5. Numerical Examples. We compare our theoretical findings with numerical experiments. Our main goal is to emphasize the influence of the right choice of approximation space.
We choose like in [F96]

$$k(x_1, \ldots, x_{2n}) \quad = \quad \frac{1}{2\zeta(6)^n \pi^n} \sum_{j_1, \ldots, j_{2n} > 0} \prod_{i=1}^{2n} \frac{\sin(j_i x_i)}{j_i^3},$$

$$u(x_1,\ldots,x_n) \quad = \quad \sum_{j_1,\ldots,j_n>0} \prod_{i=1}^{n} \frac{\sin(j_i x_i)}{j_i^3}$$

with $\zeta(6) = \sum_{i=1}^{\infty} \frac{1}{i^6} = \frac{\pi^6}{945}$ the Riemann ζ-function. The result of the integral evaluation is given by $h(x_1,\ldots,x_n) = u(x_1,\ldots,x_n)/2$. The functions k, u and h are elements of $\mathcal{H}_{mix}^{5/2-\epsilon,0}$ on T^{2n} and T^n, respectively, for all $\epsilon > 0$, i.e. the index set

$$SGA(s) := \{(\mathbf{j},\mathbf{k}) \in \mathbb{N}^{2n} : \frac{5}{2}|\mathbf{j},\mathbf{k}|_1 + \frac{5}{2}|\mathbf{k}|_1 - s|\mathbf{j}|_\infty \leq (\frac{5}{2} - s)J\} \tag{31}$$

is specially suited for the approximation of the kernel, compare (18) in subsection 3.2. For comparison we also use the index sets of the full grid

$$VG := \{(\mathbf{j},\mathbf{k}) \in \mathbb{N}^{2n} : |\mathbf{j},\mathbf{k}|_\infty \leq J\} \tag{32}$$

and the index set

$$FHP(s) \quad := \quad \{(\mathbf{j},\mathbf{k}) \in \mathbb{N}^{2n} : \frac{5}{2}|\mathbf{j},\mathbf{k}|_\infty + \frac{5}{2}|\mathbf{k}|_\infty - s|\mathbf{j}|_\infty \leq (\frac{5}{2} - s)J\}. \tag{33}$$

$FHP(s)$ is adapted to the smoothness $\mathcal{H}^{5/2}$ (for the kernel and for u), compare subsection 3.2. The index sets for the approximation of u and h are given by (21),(20). We use Algorithm 2 from section 4 together with these index sets to evaluate the integral.

Integral evaluation in 1D. We start with the case $n = 1$, that is, the kernel is two-dimensional. Figure 8 shows the accuracy of the methods that work with the above kernel- and function-approximations. The abszissa shows the number of Fourier coefficients that is used in the algorithm and the ordinate shows the error of the approximations in the $\mathcal{L}^2$ or $\mathcal{H}^1$-norm. The problem adapted index sets $SGA(0)$ for the $\mathcal{L}^2$-norm and $SGA(1)$ for the $\mathcal{H}^1$-norm give the best results. In the 1D case, the index sets $FHP(s)$ does also give good results, whereas the full grid is obviously no good choice.

Integral evaluation in 2D. Again, we compare the index sets $SGA(s)$ from (31) with the index sets VG and FHP from (32) and (33). Figure 9 visualizes the results. Now the index set $SGA(s)$ adapted to the dominating mixed smoothness yields significantly better results than the index set $FHP(s)$. Again, the full grid approximation spaces turn out to be not competitive.

6. Concluding remarks. In this paper we defined approximation spaces for the approximation of integral expressions with smooth kernels. We showed that the right choice of approximation space is crucial for good approximation results. Specifically we have seen that the smoothness of the kernel, the smoothness of u as well as the error-norm should be considered for the choice of approximation spaces. In extremal cases the influence of the dimension n on the cost of evaluating the integral expression can then be removed. In this way, we generalized results of [F95, F96] with respect to more general norms and general smoothness assumptions. Moreover we desribed blending schemes

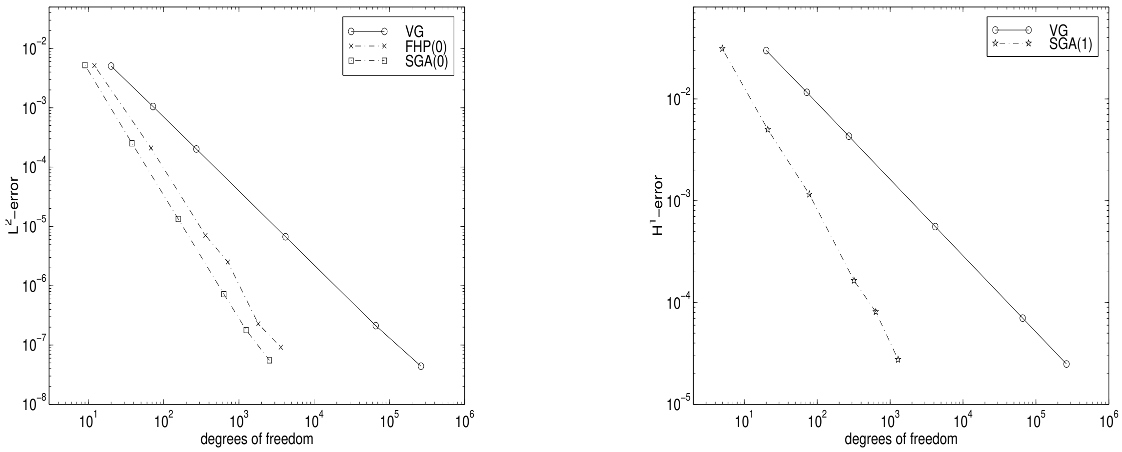

FIG. 8. *Error in $\mathcal{L}^2$- and $\mathcal{H}^1$-norm against number of coefficients used in the algorithm for different kernel approximations*

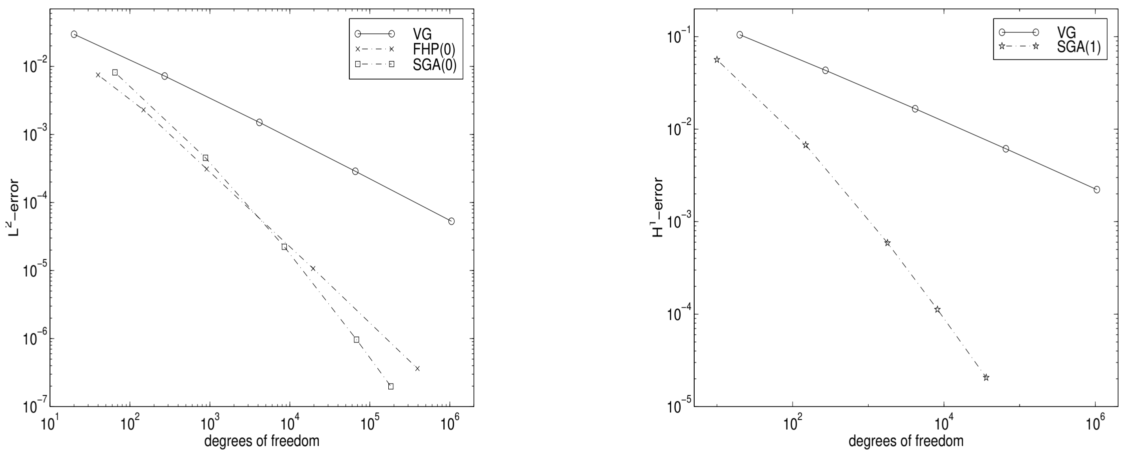

FIG. 9. *Error in $\mathcal{L}^2$- and $\mathcal{H}^1$-norm against number of coefficients used in the algorithm for different kernel approximations*

that allow the construction of efficient and dimension recursive algorithms with simple data structures.

The constructions presented in this paper are not limited to the case of trigonometric approximation but can be utilized in a similar way for a large class of multiscale basis functions, compare [Gri00]. These considerations open the way for the extension to (non periodic) boundary conditions, see [Ho00], and allow (constructive) complexity estimates for approximation problems and operator equations in Sobolev spaces.

Note that for boundary integral/element applications (pre-)wavelets have some additional advantages for compression strategies that are based on the (potential) decay properties of the kernel in physical space because they lead to a better sparsity of resulting discretized linear systems. Methods that exploit the latter decay properties are wavelet methods [Da93, Schn98] and multipol and panel clustering methods

[R85, Gre87, Hac89]. Note also that the tensor-product ansatz is not crucial for this type of compression.

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