Nonlinear Analysis 74 (2011) 3884–3895

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Generalized gradients and characterization of epi-Lipschitz sets in Riemannian manifolds

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ARTICLE INFO

Article history:
Received 17 July 2010
Accepted 14 February 2011

Keywords:
Generalized gradients
Locally Lipschitz functions
Epi-Lipschitz sets
Riemannian manifolds

ABSTRACT

In this paper, a notion of generalized gradient on Riemannian manifolds is considered and a subdifferential calculus related to this subdifferential is presented. A characterization of the tangent cone to a nonempty subset S of a Riemannian manifold M at a point x is obtained. Then, these results are applied to characterize epi-Lipschitz subsets of complete Riemannian manifolds.

1. Introduction

Nondifferentiability appears naturally in several areas of mathematics and arises explicitly in the description of various modern technological systems. Nonsmooth analysis studies the local behavior of nondifferentiable functions and sets lacking smooth boundaries.

Nondifferentiable functions are often considered on finite dimensional or infinite dimensional Banach spaces, where the linear structure plays a central role. However, in various aspects of mathematics such as control theory and matrix analysis, nonsmooth functions arise naturally on smooth manifolds; see [1,2]. Unlike a Banach space, a manifold in general does not have a linear structure and therefore new techniques are needed for dealing with nonsmooth functions defined on manifolds. In the past few years, a number of results have been obtained on numerous aspects of nonsmooth analysis and their applications on Riemannian manifolds; see, e.g. [3–7].

Generalized gradients or subdifferentials refer to several set-valued replacements for the usual derivative. These concepts are used in developing differential calculus for nonsmooth functions. The concept of the generalized gradient of a locally Lipschitz function was introduced by Clarke in 1975. This concept reduces to the classical gradient for smooth functions and the subdifferential in the sense of convex analysis for convex functions and is accompanied by a useful calculus.

Attempts have been made to replace the class of locally Lipschitz functions by classes of noncontinuous functions and develop a subdifferential calculus; see, e.g. [6,8,9] and the references therein. For lower semicontinuous functions smooth local approximations from below led to the concept of viscosity and proximal subdifferentials.

In [3] the theory of viscosity solutions of Hamilton–Jacobi equations and the corresponding calculus were extended to the setting of Riemannian manifolds (possibly of infinite dimensional). In [4,10] a notion of proximal subdifferential for functions defined on Riemannian manifolds was introduced, a calculus for nonsmooth functions on these manifolds was established and its applications were discussed. By a different approach in [2] a nonsmooth calculus on finite dimensional Riemannian manifolds was developed and its applications to Hamilton–Jacobi equations were studied.

This paper is devoted to the study of the Clarke generalized gradient for locally Lipschitz functions defined on Riemannian manifolds (either finite or infinite dimensional). This notion was introduced in [3,11,12]. We develop a basic calculus result

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0362-546X/$ – see front matter
doi:10.1016/j.na.2011.02.023

for this subdifferential. Moreover, a class of subsets of Riemannian manifolds named epi-Lipschitz, is introduced and a characterization of this class of sets, is obtained. Works dealing with this class of subsets of Euclidean spaces include those by Rockafeller [13], Cornet and Czarnecki [14,15], Czarnecki and Rifford [16]. It is worthwhile to mention that extensions of concepts concerning epi-Lipschitz subsets of Euclidean spaces to Riemannian manifolds have not been studied yet, in spite of particular importance of this class of sets which includes closed convex sets with a nonempty interior. In a sequel to the present paper we will elaborate on the applications of this class of subsets of Riemannian manifolds, providing sufficient conditions for the existence of equilibria for a class of set-valued mapping F defined on a compact epi-Lipschitz subset S of a complete parallelizable Riemannian manifold M with values in the tangent bundle. The results regarding these class of sets which will be proved are not of local type, and cannot be obtained by local techniques.

The rest of the paper is organized as follows. In Section 2, we present the definition of the generalized directional derivative on Riemannian manifolds. Then, the generalized gradient is introduced and its properties as a set-valued map are investigated. Moreover, a subdifferential calculus for locally Lipschitz functions is presented. In particular, a chain rule and Lebourg's mean value theorem are proved. Section 3 is concerned with the properties of the tangent and normal cones to closed subsets of Riemannian manifolds. Finally, Section 4 is devoted to a characterization of epi-Lipschitz subsets of Riemannian manifolds.

2. The Clarke generalized gradient

In this paper, we use the standard notations and known results of Riemannian manifolds, see, e.g. [17–19]. Let us mention some of them often used in what follows. Throughout this paper, M is a C^∞ *connected* manifold modeled on a Hilbert space H, either finite dimensional or infinite dimensional, endowed with a Riemannian metric $\langle ., .\rangle_x$ on the tangent space $T_xM \cong H$. Recall that the set S in a Riemannian manifold M is called convex if every two points $p_1, p_2 \in S$ can be joined by a unique geodesic whose image belongs to S. As usual, for a point $x \in M$, T_xM will denote the tangent space of M at x, and $\exp_x : U_x \to M$ will stand for the exponential function at x, where U_x is an open subset of T_xM. We should also recall that $\exp_x$ maps straight lines of the tangent space T_xM passing through $0_x \in T_xM$ into geodesics of M passing through x.

We will also use the parallel transport of vectors along geodesics. Recall that for a given curve $\gamma : I \to M$, number $t_0 \in I$, and a vector $V_0 \in T_{\gamma(t_0)}M$, there exists a unique parallel vector field $V(t)$ along $\gamma(t)$ such that $V(t_0) = V_0$. Moreover, the mapping defined by $V_0 \mapsto V(t_1)$ is a linear isometry between the tangent spaces $T_{\gamma(t_0)}M$ and $T_{\gamma(t_1)}M$, for each $t_1 \in I$. In the case when γ is a minimizing geodesic and $\gamma(t_0) = x$, $\gamma(t_1) = y$, we will denote this mapping by L_{xy}, and we will call it the parallel transport from T_xM to T_yM along the curve γ. Note that L_{xy} is well defined when the minimizing geodesic which connects x to y, is unique. For example, the parallel transport L_{xy} is well defined when x and y are contained in a convex neighborhood. In what follows, L_{xy} will be used wherever it is well defined.

The parallel transport allows us to measure the length of the "difference" between vectors which are in different tangent spaces. Indeed, let γ be a minimizing geodesic connecting two points $x, y \in M$, say $\gamma(t_0) = x$, $\gamma(t_1) = y$. Take vectors $v \in T_xM$, $w \in T_yM$. Then we can define the distance between v and w as the number

$$\|v - L_{yx}(w)\| = \|w - L_{xy}(v)\|.$$

The isometry L_{yx} induces another linear isometry L^*_{yx} between T_xM^* and T_yM^*, such that for every $\sigma \in T_xM^*$ and $v \in T_yM$, we have $\langle L^*_{yx}(\sigma), v\rangle = \langle \sigma, L_{yx}(v)\rangle$. We will still denote this isometry by $L_{xy} : T_xM^* \to T_yM^*$.

Recall that a real-valued function f defined on a Riemannian manifold M is said to satisfy a Lipschitz condition of rank K on a given subset S of M if $|f(x) - f(y)| \le Kd(x, y)$ for every $x, y \in S$, where d is the Riemannian distance on M. A function f is said to be Lipschitz near $x \in M$, if it satisfies the Lipschitz condition of some rank on an open neighborhood of x. A function f is said to be locally Lipschitz on M, if f is Lipschitz near x, for every $x \in M$.

Let us start with the definition of the Clarke generalized directional derivative for locally Lipschitz functions on Riemannian manifolds; see [11].

Definition 2.1. Suppose $f : M \to \mathbb{R}$ is a locally Lipschitz function on a Riemannian manifold M. Then, the generalized directional derivative of f at $x \in M$ in the direction $v \in T_xM$, denoted by $f^\circ(x; v)$, is defined as

$$f^\circ(x; v) := \limsup_{y\to x, t\downarrow 0} \frac{f \circ \varphi^{-1}(\varphi(y) + t\mathrm{d}\varphi(x)(v)) - f \circ \varphi^{-1}(\varphi(y))}{t}, \tag{2.1}$$

where (φ, U) is a chart at x.

Indeed, $f^\circ(x; v) := (f \circ \varphi^{-1})^\circ(\varphi(x); \mathrm{d}\varphi(x)(v))$. Note that this definition does not depend on charts (see [11]).

Considering $0_x \in T_xM$, we have

$$f^\circ(x; v) = (f \circ \exp_x)^\circ(0_x, v). \tag{2.2}$$

Let us introduce another equivalent definition of the generalized directional derivative for locally Lipschitz functions on Riemannian manifolds (see [3]).

Let $\phi_x : U_x \to T_xM$ be an exponential chart at x. Given another point $y \in U_x$, consider $\sigma_{y,v}(t) := \phi_y^{-1}(tw)$, a geodesic passing through y with derivative w, where (ϕ_y, y) is an exponential chart around y and $d(\phi_x o \phi_y^{-1})(0_y)(w) = v$. Then, $f^\circ(x, v)$ is defined by

$$f^\circ(x, v) = \limsup_{y\to x, t\downarrow 0} \frac{f(\sigma_{y,v}(t)) - f(y)}{t}.$$

Let us present two important examples of locally Lipschitz functions that arise naturally on Riemannian manifolds; see also [20].

Example 2.2. Let M be a Riemannian manifold and let $f : M \to (-\infty, +\infty]$ be a lower semicontinuous function that is bounded from below. For $\lambda > 0$, we define the function

$$f_\lambda(x) := \inf_{y\in M}\{f(y) + \lambda d^2(x, y)\}$$

for every $x \in M$. Then along the same lines as [8, Theorem 1.5.1] one can prove that f_λ is locally Lipschitz on M.

Example 2.3. Let $\mathrm{S}(n, \mathbb{R})$ be the linear space of symmetric $n\times n$ real matrices endowed with the Frobenius metric defined by $\langle U, V\rangle = \mathrm{tr}(UV)$. For any $A \in \mathrm{S}(n, \mathbb{R})$, let $\lambda_1, \ldots, \lambda_n$ denote the n (including repeated) real eigenvalues of A in nondecreasing order. Then for any $k \in \{1, \ldots, n\}$, $\lambda_k : \mathrm{S}(n, \mathbb{R}) \to \mathbb{R}$ is a locally Lipschitz function; see [6, p. 304]. Moreover, let Spos_n be the set of all symmetric positive definite $n \times n$ matrices with determinant 1 which is a closed submanifold of $\mathrm{S}(n, \mathbb{R})$. Then Spos_n is a Cartan–Hadamard manifold, i.e., a simply connected Riemannian manifold of nonpositive curvature (see [19, p. 334]) and for any $k \in \{1, \ldots, n\}$, $\lambda_k : \mathrm{Spos}_n \to \mathbb{R}$ is a locally Lipschitz function.

The following result is of local type and can be deduced from [8, Proposition 2.1.1].

Theorem 2.4. *Let M be a Riemannian manifold and $x \in M$. Suppose that the function $f : M \to \mathbb{R}$ is Lipschitz of rank K on an open neighborhood U of x. Then,*

(a) *for each $y \in U$ the function $v \mapsto f^\circ(y; v)$ is finite, positive homogeneous, and sub-additive on T_yM, and satisfies*

$$|f^\circ(y; v)| \le K\|v\|.$$

(b) *$f^\circ(y; v)$ is upper semicontinuous on $TM|_U$ and, as a function of v alone, is Lipschitz of rank K on T_yM, for each $y \in U$.*
(c) *$f^\circ(y; -v) = (-f)^\circ(y; v)$ for each $y \in U$ and $v \in T_yM$.*

The generalized gradient or the Clarke subdifferential of a locally Lipschitz function f at $x \in M$, denoted by $\partial f(x)$, is the subset of T_xM^* whose support function is $f^\circ(x; .)$. Thus $\xi \in \partial f(x)$ if and only if $f^\circ(x; v) \ge \langle\xi, v\rangle$ for all v in T_xM. As a consequence of the definition of the generalized directional derivative, we conclude the following proposition; see [11].

Proposition 2.5. *Let M be a Riemannian manifold and $x \in M$. Suppose that $f : M \longrightarrow \mathbb{R}$ is Lipschitz near x and (φ, U) is a chart at x. Then*

$$\partial f(x) = \mathrm{d}\varphi(x)^*[\partial(f \circ \varphi^{-1})(\varphi(x))], \tag{2.3}$$

where $*$ denotes the adjoint.

Therefore, we have $\partial f(x) = \partial(f \circ \exp_x)(0_x)$.

Example 2.6. Let M be a finite dimensional manifold and $f_1, \ldots, f_k : M \to \mathbb{R}$ be C^1 functions. Then $f : M \to \mathbb{R}$ defined by $f(x) := \max_{1\le i\le l} \min_{j\in M_i} f_j(x)$, where for every $1 \le i \le l$, $M_i \subseteq \{1, \ldots, k\}$, is a locally Lipschitz function. Also, for every $v \in T_xM$, $f^\circ(x, v) = \max_{i\in\tilde{I}(f(x))} \min_{j\in\tilde{J}_i(f(x))} \langle df_j(x), v\rangle$, where $\tilde{J}_i(v) = \{j \in M_i : \min_{p\in M_i} v_p = v_j\}$ and $\tilde{I}(v) = \{i \in \{1, \ldots, k\} : \tilde{J}_i(v) \ne \emptyset, \min_{p\in\tilde{J}_i(v)} v_p = \max_{1\le i\le l} \min_{j\in M_i} v_j\}$. Moreover, $\partial f(x) = \mathrm{co}\{\mathrm{d}f_j(x) : j \in \bar{I}(x)\}$, where $\bar{I}(x) = \{i : x \in \overline{\mathrm{int}\{x : f(x) = f_i(x)\}}\}$.

We proceed now to derive some of the basic properties of the generalized gradient. A set-valued function $F : X \rightrightarrows Y$, where X, Y are topological spaces, is said to be upper semicontinuous at x, if for every open neighborhood U of $F(x)$ there exists an open neighborhood V of x such that

$$y \in V \Longrightarrow F(y) \subseteq U.$$

A set-valued function $F : X \rightrightarrows Y$, where X is a topological space and Y is a Hilbert space, is said to be upper hemicontinuous at x_0 if for every $y^* \in Y^*$, the function $x \mapsto \sigma(F(x), y^*)$ is upper semicontinuous at x_0, where $\sigma(F(x), y^*)$ is support function of the set $F(x)$.

Lemma 2.7. *Let M be an n-dimensional manifold. Consider the set-valued function $G : M \rightrightarrows TM^*$ such that $G(x) \subseteq T_xM^*$ for every $x \in M$. Suppose in a chart (ψ, W) at $x \in M$, G is represented by*

$$G(y) = \left\{ \sum_{i=1}^{n} g_i(y) \mathrm{d}x_i|_y : \{\mathrm{d}x_i|_y\} \text{ is a local basis of } T_yM^* \text{ in the chart } (\psi, W) \right\}.$$

Then, G is an upper semicontinuous function at x if and only if $g : W \rightrightarrows \mathbb{R}^n$ defined by $g(y) := \{(g_1(y), g_2(y), \dots, g_n(y))\}$ is upper semicontinuous at x.

Proof. Let $G : M \rightrightarrows TM^*$ be upper semicontinuous at x and $(\widetilde{\psi}, \widetilde{W})$ be the chart of TM^* such that $\widetilde{W} = \pi^{-1}(W)$ and

$$\widetilde{\psi} : \pi^{-1}(W) \longrightarrow \psi(W) \times \mathbb{R}^n$$
$$\widetilde{\psi}(X_y^*) := (\psi(y), a_1(y), \dots, a_n(y)),$$

where $X_y^* = \sum_{i=1}^{n} a_i(y) \mathrm{d}x_i|_y$. Suppose $N \subseteq \mathbb{R}^n$ is an open neighborhood of $g(x)$. Then, $\widetilde{\psi}^{-1}(\psi(W) \times N) \subseteq TM^*$ is open and $G(x) \subseteq \widetilde{\psi}^{-1}(\psi(W) \times N)$. Hence there exists an open neighborhood $V(x) \subseteq M$ such that for every $y \in V(x)$, we have $G(y) \subseteq \widetilde{\psi}^{-1}(\psi(W) \times N)$. Thus, for $y \in V(x)$ we have that $g(y) \subseteq N$.

Conversely, let U be an open subset of TM^* and $G(x) \subseteq U \subseteq TM^*$. Since $G(x) \subseteq T_xM^*$ and $T_xM^* \subseteq \pi^{-1}(W)$, it follows that $U \cap \pi^{-1}(W)$ is nonempty and open. Thus $\pi_2(\widetilde{\psi}(U \cap \pi^{-1}(W))) \subseteq \mathbb{R}^n$ is open, where π_2 is the projection function on the second coordinate. Therefore, there exists an open neighborhood $V(x) \subseteq M$, such that if $y \in V(x)$ then $g(y) \subseteq \pi_2(\widetilde{\psi}(U \cap \pi^{-1}(W)))$. Hence, $G(y) \subseteq U$. □

Remark 2.8. Let M be a Riemannian manifold.

(a) An easy consequence of the definition of the parallel translation along a curve as a solution to an ordinary linear differential equation, implies that the mapping

$$C : TM^* \to T_{x_0}M^*, \qquad C(x, \xi) = L_{xx_0}(\xi),$$

is continuous at (x_0, ξ_0), that is, if $(x_n, \xi_n) \to (x_0, \xi_0)$ in TM^* then $L_{x_n x_0}(\xi_n) \to L_{x_0 x_0}(\xi_0) = \xi_0$, for every $(x_0, \xi_0) \in TM^*$; see [3, Remark 6.11].

(b) By the continuity properties of the parallel transport and the geodesic, see [4, Theorem 35], for fixed point $z \in M$ and for each $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$\|L_{xy}L_{zx} - L_{zy}\| \le \varepsilon \quad \text{provided that } d(x, y) < \delta.$$

(c) Utilizing the properties of the exponential map on Riemannian manifold M, for fixed $x \in M$ and for each $\varepsilon > 0$, we may find number $\delta_x > 0$ such that

$$\|d(\exp_x^{-1})(y) - L_{yx}\| \le \varepsilon \quad \text{provided that } d(x, y) < \delta_x.$$

Theorem 2.9. *Let M be a Riemannian manifold, $x \in M$ and $f : M \to \mathbb{R}$ be a Lipschitz function of rank K near x. Then,*

(a) *$\partial f(x)$ is a nonempty, convex,* weak*-*compact subset of T_xM^*, and $\|\xi\|_* \le K$ for every $\xi \in \partial f(x)$.*

(b) *For every v in T_xM, we have*

$$f^\circ(x; v) = \max\{\langle \xi, v\rangle : \xi \in \partial f(x)\}.$$

(c) *If $\{x_i\}$ and $\{\xi_i\}$ are sequences in M and TM^* such that $\xi_i \in \partial f(x_i)$ for each i, and if $\{x_i\}$ converges to x and ξ is a* weak*-*cluster point of the sequence $\{L_{x_i x}(\xi_i)\}$, then we have $\xi \in \partial f(x)$.*

(d) *If M is finite dimensional, then ∂f is upper semicontinuous at x.*

Proof. Properties (a) and (b) are easily shown to be true. Let us prove (c). Fix $v \in T_xM$. For each i, we have $f^\circ(x_i; L_{xx_i}(v)) \ge \langle \xi_i, L_{xx_i}(v)\rangle$. The sequence $\{\langle \xi_i, L_{xx_i}(v)\rangle\} = \{\langle L_{x_i x}(\xi_i), v\rangle\}$ is bounded in $\mathbb{R}$, and contains terms that are arbitrarily near $\langle \xi, v\rangle$. Let us extract a subsequence of $\{L_{x_i x}(\xi_i)\}$ (without relabeling) such that $\{\langle L_{x_i x}(\xi_i), v\rangle\} = \{\langle \xi_i, L_{xx_i}(v)\rangle\} \to \langle \xi, v\rangle$. By Remark 2.8(a) we have that $L_{xx_i}(v) \to v$. Since f° is upper semicontinuous in (x, v), it follows that passing to the limit in the preceding inequality gives $f^\circ(x; v) \ge \langle \xi, v\rangle$. Since v is arbitrary, we conclude $\xi \in \partial f(x)$.

We turn now to (d). Let M be an n-dimensional manifold. For each $z \in U$, we define

$$T(z) := \partial(f \circ \varphi^{-1})(\varphi(z)),$$

where (U, φ) is a chart at x. The function

$$F : \varphi(U) \rightrightarrows \mathbb{R}^n,$$

defined by $F(y) := \partial(f \circ \varphi^{-1})(y)$ is upper semicontinuous and $T(z) = F \circ \varphi(z)$. Thus, T is upper semicontinuous. On the other hand,

$$\partial f(x) = \mathrm{d}\varphi(x)^*[\partial(f \circ \varphi^{-1})(\varphi(x))],$$

and

$$\mathrm{d}\varphi(x)^*\left[\sum_{i=1}^{n}\pi_i(\partial(f\circ\varphi^{-1})(\varphi(x)))e_i\right]=\sum_{i=1}^{n}\pi_i(\partial(f\circ\varphi^{-1})(\varphi(x)))\mathrm{d}\varphi(x)^*(e_i),$$

where $\{e_i\}$ is a basis of $\mathbb{R}^n$ and π_i is the projection function on ith coordinate. Since $\{\mathrm{d}\varphi(y)^*(e_i)\}$ is a local basis of T_yM^* in the chart (U,φ) and T is upper semicontinuous at x, it follows from Lemma 2.7 that ∂f is upper semicontinuous at x. □

Theorem 2.10. *Let M be a finite dimensional Riemannian manifold and $x_0\in M$. Suppose that $f:M\to\mathbb{R}$ is a Lipschitz function of rank K on a geodesic ball $B_r(x_0)$. Then for each $\varepsilon>0$, there exists $\delta>0$ such that for $x\in B_\delta(x_0)$*

$$L_{xx_0}(\partial f(x))\subseteq\partial f(x_0)+\varepsilon B_{T_{x_0}M^*},$$

where $B_{T_{x_0}M^*}$ is the unit ball of $T_{x_0}M^*$.

Proof. For $\varepsilon>0$ the set $\partial f(x_0)+\varepsilon B_{T_{x_0}M^*}$ is an open neighborhood of $L_{x_0x_0}(\partial f(x_0))=\partial f(x_0)$. It follows from Remark 2.8 that there exists an open neighborhood $V\subset TM^*$ of $\partial f(x_0)$ such that

$$C(V)\subseteq\partial f(x_0)+\varepsilon B_{T_{x_0}M^*}.$$

By the upper semicontinuity of ∂f, there exists a neighborhood V' of x_0 such that for each $x\in V'$, we have $\partial f(x)\subseteq V$. Now let $x\in V'$, then

$$L_{xx_0}(\partial f(x))\subseteq\partial f(x_0)+\varepsilon B_{T_{x_0}M^*},$$

and the proof is complete. □

The previous theorem shows that for fixed point $x_0\in M$, the map $x\rightrightarrows L_{xx_0}(\partial f(x))$ is an upper semicontinuous set-valued map at x_0. Therefore, it is upper hemicontinuous at x_0; see [21, section 9.2]. In the following theorem, we will prove that, when M is a finite dimensional Riemannian manifold, for every $y\in M$ and every geodesic ball $B_r(y)$ around y, the map $x\rightrightarrows L_{xy}(\partial f(x))$ is upper hemicontinuous on $B_r(y)$.

Theorem 2.11. *Let M be a finite dimensional Riemannian manifold and let $f:M\to\mathbb{R}$ be a locally Lipschitz function. Suppose that $(y,v)\in TM$ and $B_r(y)$ is a geodesic ball around y. Then the function $x\mapsto\sigma(L_{xy}(\partial f(x)),v)$ is upper semicontinuous on $B_r(y)$, where σ is the support function of the set $L_{xy}(\partial f(x))$.*

Proof. Let $\bar{x}\in B_r(y)$. We prove that for each $\varepsilon>0$ there exists $\delta>0$ such that for every $x\in B_r(y)$ satisfying $d(x,\bar{x})<\delta$ we have

$$\sigma(L_{xy}(\partial f(x)),v)\le\sigma(L_{\bar{x}y}(\partial f(\bar{x})),v)+\varepsilon.$$

By Theorem 2.10, for each $\varepsilon>0$ there exists $\delta_1>0$ such that if $d(x,\bar{x})<\delta_1$, then

$$L_{x\bar{x}}(\partial f(x))\subseteq\partial f(\bar{x})+\varepsilon B_{T_{\bar{x}}M^*},$$

where $B_{T_{\bar{x}}M^*}$ is the unit ball of the $T_{\bar{x}}M^*$. Hence,

$$L_{\bar{x}y}(L_{x\bar{x}}(\partial f(x)))\subseteq L_{\bar{x}y}(\partial f(\bar{x})+\varepsilon B_{T_{\bar{x}}M^*}).$$

By Remark 2.8, we can find $\delta_2>0$ such that if $d(x,\bar{x})<\delta_2$, then

$$\|L_{\bar{x}y}L_{x\bar{x}}-L_{xy}\|\le\varepsilon.$$

Therefore, for each x satisfying $d(x,\bar{x})<\min\{\delta_1,\delta_2\}$ and for each $\xi\in\partial f(x)$ we have that

$$|\langle v,L_{xy}(\xi)-L_{\bar{x}y}(L_{x\bar{x}}(\xi))\rangle|\le\|v\|\,\|\xi\|\,\|L_{\bar{x}y}L_{x\bar{x}}-L_{xy}\|\le\varepsilon K\|v\|,$$

where K is the Lipschitz constant of f near $\bar{x}$. It follows that

$$\begin{aligned}\sup_{\xi\in\partial f(x)}\langle v,L_{xy}(\xi)\rangle&\le\sup_{\xi\in\partial f(x)}\langle v,L_{\bar{x}y}(L_{x\bar{x}}(\xi))\rangle+\|v\|\varepsilon K\\&\le\sup_{\xi\in\partial f(\bar{x})}\langle v,L_{\bar{x}y}(\xi)\rangle+\varepsilon\|v\|+\varepsilon\|v\|K.\end{aligned}$$

Thus the proof is complete. □

Remark 2.12. Obviously, by the previous theorem for every $\gamma\in\mathbb{R}$ and $(y,v)\in TM$,

$$\{x\in B_r(y):\gamma<\inf_{\xi\in\partial f(x)}\langle L_{yx}(v),\xi\rangle\},\tag{2.4}$$

is an open subset of M, where $B_r(y)$ is a geodesic ball around y.

Note that in Theorem 2.10, M is a finite dimensional Riemannian manifold. The following useful result is a weak version of Theorem 2.10 on infinite dimensional Riemannian manifolds.

Theorem 2.13. *Let M be a Riemannian manifold, $x_0 \in M, f : M \to \mathbb{R}$ be Lipschitz on geodesic ball $B_r(x_0)$ of rank K. Then for every $v \in T_{x_0}M$ and $\varepsilon > 0$ there exists δ_v such that for each $x \in B_{\delta_v}(x_0)$ and $w \in \partial f(x)$, there is a $w_0 \in \partial f(x_0)$ satisfying*

$$|\langle L_{xx_0}(w) - w_0, v\rangle| < \varepsilon.$$

Proof. We proceed by contradiction. There would be an element $v \in T_{x_0}M$ and a number $\varepsilon > 0$ and sequences $\{x_n\} \subset M$, $\{\xi_n\} \subset \partial f(x_n)$ such that $x_n \to x_0$. We may assume that for each n, $x_n \in B_r(x_0)$. Hence, for each $w \in \partial f(x_0)$ we have that

$$|\langle L_{x_n x_0}(\xi_n) - w, v\rangle| > \varepsilon. \tag{2.5}$$

Since $\|\xi_n\|_{T_{x_n}M^*} \leq K$, it follows that $\|L_{x_n x_0}(\xi_n)\|_{T_{x_0}M^*} \leq K$. Therefore there is a subsequence $L_{x_{n_i}x_0}(\xi_{n_i}) \to \xi_0$ in weak* topology. Clearly, $\xi_0 \in \partial f(x_0)$ and if we replace w by ξ_0 in (2.5), we get a contradiction. □

3. Subdifferential calculus

In this section, we present a subdifferential calculus for locally Lipschitz functions defined on Riemannian manifolds. An easy consequence of the definition of the generalized gradient is the following proposition.

Proposition 3.1. *Let M be a Riemannian manifold (possibly of infinite dimension). Then the following assertions hold:*

(a) *if $f : M \to \mathbb{R}$ is Lipschitz near x, then for each scalar λ, we have that $\partial(\lambda f) = \lambda\partial f(x)$.*
(b) *If $f_i : M \to \mathbb{R}$ $(i = 1, 2, \ldots, n)$ are Lipschitz near x and λ_i $(i = 1, 2\ldots, n)$ are scalars. Then $f := \sum_{i=1}^n \lambda_i f_i$ is Lipschitz near x, and we have*

$$\partial\left(\sum_{i=1}^n \lambda_i f_i\right)(x) \subset \sum_{i=1}^n \lambda_i \partial f_i(x).$$

Theorem 3.2 (*The Chain Rule*)**.** *Let M be a Riemannian manifold (possibly of infinite dimensional) and N be a finite dimensional Riemannian manifold. Suppose that $F : M \longrightarrow N$ is Lipschitz near x and $g : N \longrightarrow \mathbb{R}$ is Lipschitz near $F(x)$. Then $f(y) = g(F(y))$ is Lipschitz near x, and we have*

$$\partial f(x) \subseteq \overline{co^{w^*}}\{\partial(\langle \gamma, \exp_{F(x)}^{-1} \circ F(.)\rangle)(x) : \gamma \in \partial g(F(x))\},$$

where $\overline{co^{w^*}}$ signifies the weak*-*closed convex hull.*

Proof. Let $\exp_{F(x)} : \widetilde{U(0)} \subseteq T_{F(x)}N \to V(F(x))$ be a diffeomorphism. Since F is Lipschitz near x, there exists an open neighborhood $W(x)$ such that $F(W(x)) \subseteq V(F(x))$. Define $\widetilde{g} : g \circ \exp_{F(x)} := \widetilde{U(0)} \subseteq T_{F(x)}N \longrightarrow \mathbb{R}$ and $\widetilde{F} := \exp_{F(x)}^{-1} \circ F : W(x) \longrightarrow \widetilde{U(0)} \subseteq T_{F(x)}N \simeq \mathbb{R}^n$. We claim that

$$\partial f(x) \subseteq \overline{co^{w^*}}\{\partial\langle \gamma, \widetilde{F}(.)\rangle(x) : \gamma \in \partial\widetilde{g}(\widetilde{F}(x))\}.$$

To prove the claim, let (φ, U) be a chart of M at x such that $U \subseteq W$. Then $\widetilde{F} \circ \varphi^{-1}$ is Lipschitz near $\varphi(x)$ and

$$\partial(f \circ \varphi^{-1})(\varphi(x)) \subseteq \overline{co^{w^*}}\{\partial\langle \gamma, \widetilde{F} \circ \varphi^{-1}(.)\rangle(\varphi(x)) : \gamma \in \partial\widetilde{g}(\widetilde{F} \circ \varphi^{-1}(\varphi(x))) = \partial\widetilde{g}(\widetilde{F}(x))\}.$$

Now, for $\xi \in \partial f(x)$ we have that $\xi = \mathrm{d}\varphi(x)^* q$ where $q \in \partial(f \circ \varphi^{-1})(\varphi(x))$. Thus $q = \lim_j \sum_{i=1}^n t_{j,i} q_{j,i}$ where $q_{j,i} \in \langle \gamma_i, \widetilde{F} \circ \varphi^{-1}(.)\rangle(\varphi(x))$ and $\gamma_i \in \partial\widetilde{g}(\widetilde{F} \circ \varphi^{-1}(\varphi(x))) = \partial\widetilde{g}(\widetilde{F}(x))$. It follows that

$$\begin{aligned}\mathrm{d}\varphi(x)^* q_{j,i} \in \partial\langle \gamma_i, \widetilde{F}(.)\rangle(x) &= \mathrm{d}\varphi(x)^*(\partial(\langle \gamma_i, \widetilde{F}(.)\rangle \circ \varphi^{-1})(\varphi(x))) \\ &= \mathrm{d}\varphi(x)^*(\partial(\langle \gamma_i, \widetilde{F}) \circ \varphi^{-1}(.)\rangle(\varphi(x))).\end{aligned}$$

Therefore, $\xi = \mathrm{d}\varphi(x)^*(\lim_j \sum_{i=1}^n t_{j,i} q_{j,i}) = \lim_j \sum_{i=1}^n t_{j,i} \xi_{j,i}$ where $\xi_{j,i} \in \partial\langle \gamma_i, \widetilde{F}(.)\rangle(x)$ and $\gamma_i \in \partial\widetilde{g}(\widetilde{F}(x))$. Hence

$$\xi \in \overline{co^{w^*}}\{\partial\langle \gamma, \widetilde{F}(.)\rangle(x) : \gamma \in \partial\widetilde{g}(\widetilde{F}(x))\},$$

and the proof is complete. □

Theorem 3.3 (*Lebourg's Mean Value Theorem*)**.** *Let M be a finite dimensional Riemannian manifold, $x, y \in M$ and $\gamma : [0, 1] \longrightarrow M$ be a smooth path joining x and y. Let f be a Lipschitz function around $\gamma[0, 1]$. Then, there exist $0 < t_0 < 1$ and $\xi \in \partial f(\gamma(t_0))$ such that*

$$f(y) - f(x) = \langle \xi, \gamma'(t_0)\rangle.$$

Proof. Consider the function $\varphi : [0,1] \longrightarrow \mathbb{R}$ defined by

$$\varphi(t) := f(\gamma(t)) - G(t),$$

where

$$G(t) := tf(y) + (1-t)f(x).$$

The function φ is continuous and $\varphi(0) = \varphi(1) = 0$. The interval $[0,1]$ is compact and φ attains its maximum and minimum in closed interval $[0,1]$. If both maximum and minimum are boundary points, then $\varphi \equiv 0$. Hence, for each $t \in (0,1)$ we have $0 \in \partial\varphi(t)$. If maximum or minimum is an interior point, then there exists $t_0 \in (0,1)$ such that $0 \in \partial\varphi(t_0)$. The function $G(t)$ is of class C^2 and hence according to Proposition 3.1, we have that

$$0 \in \partial\varphi(t_0) \subseteq \partial(f(\gamma(t_0))) - \partial G(t_0).$$

Thus

$$f(y) - f(x) = G'(t_0) \in \partial(f(\gamma(t_0))).$$

It follows from Theorem 3.2 that

$$\begin{aligned} f(y) - f(x) &= G'(t_0) \in \overline{co^{w^*}}\{\partial\langle \xi, \exp^{-1}_{\gamma(t_0)} \circ \gamma(.)\rangle(t_0) : \xi \in \partial f(\gamma(t_0))\} \\ &= \overline{co^{w^*}}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\langle \xi, \exp^{-1}_{\gamma(t_0)} \circ \gamma(.)\rangle(t)|_{t=t_0} : \xi \in \partial f(\gamma(t_0))\right\} \\ &= \{\langle \xi, \gamma'(t_0)\rangle : \xi \in \partial f(\gamma(t_0))\}. \quad \square \end{aligned}$$

We conclude this section with the following proposition which can be deduced from [8, Theorem 2.3.2].

Proposition 3.4. *Let M and N be two Riemannian manifolds, $F : M \longrightarrow N$ be continuously differentiable near x and $g : N \longrightarrow \mathbb{R}$ be Lipschitz near $F(x)$. Then $f := g \circ F$ is Lipschitz near x and we have*

$$\partial f(x) \subseteq \mathrm{d}F(x)^* \partial g(F(x)).$$

If $\mathrm{d}F(x) : T_xM \longrightarrow T_{F(x)}N$ is onto, then the equality holds.

4. Tangents and normals

Let S be a nonempty closed subset of a Riemannian manifold M. We define $d_S : M \longrightarrow \mathbb{R}$ by

$$d_S(x) := \inf\{d(x,s) : s \in S\},$$

where d is the Riemannian distance on M. It is obvious that d_S is Lipschitz of rank 1. If S is convex, then d_S is convex too.

Definition 4.1. Let S be a nonempty closed subset of Riemannian manifold M, $x \in S$ and (φ, U) be a chart of M at x. Then the (Clarke) tangent cone to S at x, denoted by $T_S(x)$ is defined as follows:

$$T_S(x) := \mathrm{d}\varphi(x)^{-1}[T_{\varphi(S\cap U)}(\varphi(x))],$$

where $T_{\varphi(S\cap U)}(\varphi(x))$ is tangent cone to $\varphi(S \cap U)$ as a subset of the Hilbert space H at $\varphi(x)$.

Obviously, $0_x \in T_S(x)$ and $T_S(x)$ is closed and convex.

Remark 4.2. The definition of $T_S(x)$ does not depend on the choice of the chart φ at x; see [11, Lemma 3.4]. Hence, for any normal neighborhood U of x, we have that

$$T_S(x) = T_{\exp_x^{-1}(S\cap U)}(0_x). \tag{4.1}$$

It follows from [8, Proposition 2.5.2] that $v \in T_S(x)$ if and only if for every normal neighborhood U of x and every sequence $(z_i) \subset \exp_x^{-1}(S \cap U)$ converging to 0_x and sequence t_i in $(0,\infty)$ decreasing to 0, there exists a sequence $(v_i) \subset T_xM$ converging to v such that for all i, $z_i + t_i v_i \in \exp_x^{-1}(S \cap U)$.

In the case of submanifolds of $\mathbb{R}^n$, the tangent space and the normal space are orthogonal to one another. In an analogous manner, for a closed subset S of a Riemannian manifold, the normal cone to S at x, denoted $N_S(x)$, is defined as the (negative) polar of the tangent cone $T_S(x)$, i.e.

$$N_S(x) := T_S(x)^\circ := \{\xi \in T_xM^* : \langle \xi, z\rangle \le 0\ \forall z \in T_S(x)\}.$$

An easy consequence of this definition is the following proposition.

Proposition 4.3. (a) *$N_S(x)$ is a* weak**-closed convex cone.*
(b) *$N_S(x) = \mathrm{d}\varphi(x)^*(N_{\varphi(S\cap U)}(\varphi(x)))$, where $N_{\varphi(S\cap U)}(\varphi(x))$ is normal cone to $\varphi(S \cap U)$ as a subset of the Hilbert space H at $\varphi(x)$.*

The next theorem shows that, analogous to the case of Banach spaces, we have a characterization of the tangent cone on manifolds. Let us invoke the concept of proximal subdifferential of lower semicontinuous functions defined on Riemannian manifolds. See [4,10] for the details.

Definition 4.4. Let M be a Riemannian manifold, $x \in M$ and $f : M \to (-\infty, +\infty]$ be a lower semicontinuous function. The proximal subdifferential of f at x, denoted by $\partial_P f(x)$, is defined as $\partial_P (f \circ \exp_x)(0_x)$.

As a consequence of the definition of $\partial_P (f \circ \exp_x)(0_x)$ one has $\xi \in \partial_P f(x)$ if and only if there is $\sigma > 0$ such that

$$f(y) \geq f(x) + \langle \xi, \exp_x^{-1}(y)\rangle - \sigma d(x, y)^2 \tag{4.2}$$

for every y in a neighborhood of x.

Now, if S is a closed subset of Riemannian manifold M, $x \in S$. We define the proximal normal cone to S at x, denoted by $N_S^P(x)$, as $N^P_{\exp_x^{-1}(U \cap S)}(0_x)$, where U is any normal neighborhood of x.

The following lemma is an easy consequence of the definition of $N_S^P(x)$.

Lemma 4.5. *Let S be a closed subset of Riemannian manifold M, $x \in S$. Then $\xi \in N_S^P(x)$ if and only if there is $\sigma > 0$ such that*

$$\langle \xi, \exp_x^{-1}(y)\rangle \leq \sigma d(y, x)^2,$$

for every y in a neighborhood of x. Moreover, $\partial_P d_S(x) \subset N_S^P(x)$.

Remark 4.6. It is easy to verify that $N_S^P(x) = \partial_P \delta_S(x)$; here δ_S is the indicator function of S defined by $\delta_S(x) = 0$ if $x \in S$ and $\delta_S(x) = \infty$ if $x \notin S$.

Note that as we mentioned earlier throughout this paper all manifolds are assumed to be connected.

Lemma 4.7. *Let S be a nonempty closed subset of a complete finite dimensional Riemannian manifold M (or else, an infinite dimensional manifold M with the property that every two points of M are connected by a minimizing geodesic) and $x \in M - S$. If $\partial_P d_S(x) \neq \emptyset$, then d_S is differentiable at x. Moreover, there is an $s_0 \in S$ such that*

(a) *every minimizing sequence of $d_S(x)$ converges to s_0;*
(b) *$d_S(x) = d(x, s_0)$ and $d(x, s) > d_S(x)$ for every $s \in S$, $s \neq s_0$;*
(c) *there is a unique minimizing geodesic joining x and s_0;*
(d) *$L_{xs_0}(\partial_P d_S(x)) \subset N_S^P(s_0)$.*

Proof. The differentiability of d_S at x and the existence of $s_0 \in S$ satisfying assertions (a)–(c) are proved in [10, Theorem 3.5]. To prove (d), let $\xi \in \partial_P d_S(x)$. Then using (4.2) one has $2d_S(x)\xi \in \partial_P d_S^2(x)$. On the other hand by [10, Theorem 3.3], d_S^2 is differentiable at x and its differential at x is $2d_S(x)\frac{\partial d}{\partial x}(x, \hat{x})$ where $\hat{x}$ is on the unique minimizing geodesic connecting x and s_0 and close enough to x. Hence, $\xi = \frac{\partial d}{\partial x}(x, \hat{x})$. Since the function $g(s) = (d(s, \hat{y}) + d(\hat{y}, x))^2$, where $\hat{y}$ is on the unique minimizing geodesic connecting x and s_0 and close enough to s_0, attains a minimum at s_0 on S, it follows that $0 \in 2d(x, s_0)\frac{\partial d}{\partial y}(\hat{y}, s_0) + \partial_P \delta_S(s_0)$. Therefore by Remark 4.6 and [10, Theorem 3.3], $L_{xs_0}(\xi) \in N_S^P(s_0)$. $\square$

Now, we can deduce the key results that can help us arrive at our goal. The proof of the following lemma is straightforward.

Lemma 4.8. *Let M be a Riemannian manifold.*

(a) *If f is Lipschitz near x, then*

$$\partial f(x) = \overline{\mathrm{co}}\{w - \lim_{i\to\infty} \xi_i : \xi_i \in \partial_P f(x_i), x_i \to x\}.$$

(b) *If S is a closed subset of M containing x, then*

$$N_S(x) = \overline{\mathrm{co}}\{w - \lim_{i\to\infty} \xi_i : \xi_i \in N_S^P(x_i), x_i \to x\},$$

where $w - \lim$ signifies weak limit.

Lemma 4.9. *Let S be a nonempty closed subset of a complete finite dimensional Riemannian manifold M (or else, an infinite dimensional manifold M with the property that every two points of M are connected by a minimizing geodesic) and $x \in S$. Then $N_S(x) = \overline{\{\bigcup_{\lambda \geq 0} \lambda \partial d_S(x)\}}$.*

Proof. We just consider the infinite dimensional case. Suppose $\xi \in N_S^P(x)$. By using exact penalization on Riemannian manifolds, for any $\varepsilon > 0$ we have $\xi/(\|\xi\| + \varepsilon) \in \partial_P d_S(x)$. Then we can deduce that

$$B := \{w - \lim_{i\to\infty} \xi_i : \xi_i \in N_S^P(x_i), x_i \to x\} \subseteq \bigcup_{\lambda\geq 0} \lambda\{w - \lim_{i\to\infty} \xi_i : \xi_i \in \partial_P d_S(x_i), x_i \to x\}.$$

Lemma 4.8 implies $N_S(x) \subseteq \overline{\{\bigcup_{\lambda\geq 0} \lambda\partial d_S(x)\}}$.

To prove the reverse inclusion, we claim that the set $A := \{w - \lim_{i\to\infty} \xi_i : \xi_i \in \partial_P d_S(x_i), x_i \to x\}$ is contained in the set $B := \{w - \lim_{i\to\infty} \xi_i : \xi_i \in N_S^P(x_i), x_i \to x\}$. Then the assertion follows from Lemma 4.8.

Let $\xi = w - \lim_{i\to\infty} \xi_i$ such that $\xi_i \in \partial_P d_S(x_i)$ and $\{x_i\}$ converges to x. If $\{x_i\}$ has a subsequence in S, we extract this subsequence without relabeling. It follows from Lemma 4.5 that $\xi_i \in N_S^P(x_i)$. Otherwise, $\{x_i\}$ has a subsequence in $M - S$ and we extract this subsequence without relabeling. By Lemma 4.7(b) and (d) there exists a sequence $\{s_i\} \subset S$ such that for every $s \in S$, $d(s_i, x_i) \leq d(s, x_i)$ and $L_{x_i s_i}(\xi_i) \in N_S^P(s_i)$. Clearly, s_i converges to x. Thus $\xi \in B$, which ends the proof of the claim. □

Theorem 4.10. *Let S be a closed subset of a Riemannian manifold M, $x \in S$ and $v \in T_xM$. The following assertions hold.*

(i) *If $d_S^\circ(x, v) = 0$, then $v \in T_S(x)$.*
(ii) *Conversely, if in addition M is complete finite dimensional (or else, an infinite dimensional manifold with the property that every two points of M are connected by a minimizing geodesic) and $v \in T_S(x)$, then $d_S^\circ(x, v) = 0$.*

Proof. To prove assertion (i), let U be a normal neighborhood of x and suppose that the sequences $(z_i) \subset \exp_x^{-1}(S\cap U)$, $z_i \to 0_x$ and $t_i \downarrow 0$ are given. By a well-known property of the exponential map there exists a $\delta > 0$ such that $\exp_x$ is bi-Lipschitz C^∞ diffeomorphism on $B(0_x, \delta)$. We may assume that the geodesic ball $\tilde{U} := B(x, \delta)$ is contained in U. Since $z_i + t_i v \to 0_x$, it follows that for all i large enough $z_i + t_i v \in B(0_x, \delta)$. On the other hand $d_S(\exp_x(z_i)) = 0$ and by hypothesis $d_S^\circ(x, v) = 0$, hence

$$\lim_{i\to 0} \frac{(d_S \circ \exp_x)(z_i + t_i v)}{t_i} = 0. \tag{4.3}$$

For each i, we choose $s_i \in S$ such that

$$d(s_i, \exp_x(z_i + t_i v)) \leq d_S(\exp_x(z_i + t_i v)) + \frac{t_i}{i}. \tag{4.4}$$

Therefore, $s_i \to x$ and for all i large enough $s_i \in \tilde{U}$. Since the map $\exp_x^{-1}$ is Lipschitz on $\tilde{U}$, there exists a number $C > 0$ such that for all i large enough

$$\frac{1}{C}\| \exp_x^{-1}(s_i) - z_i - t_i v\| \leq d(s_i, \exp_x(z_i + t_i v)).$$

It follows from (4.4) that

$$\| \exp_x^{-1}(s_i) - z_i - t_i v\| \leq C d_S(\exp_x(z_i + t_i v)) + C\frac{t_i}{i}. \tag{4.5}$$

If we set $v_i := \frac{\exp_x^{-1}(s_i) - z_i}{t_i}$, then by (4.3) and (4.5), $v_i \to v$. Moreover, for all i large enough $\exp_x(t_i v_i + z_i) = s_i \in S \cap U$.

To prove (ii), let $v \in T_S(x)$ and ξ be an arbitrary element of $\partial d_S(x)$. Then Lemma 4.9 implies $\xi \in N_S(x)$ and by polarity we get $\langle \xi, v\rangle \leq 0$. Thus $d_S^\circ(x, v) \leq 0$, as required. □

The following theorem and its corollary will be fundamental to the next section of this paper.

Theorem 4.11. *Let M be a complete finite dimensional Riemannian manifold (or else, an infinite dimensional manifold with the property that every two points of M are connected by a minimizing geodesic), $x \in M$, $f : M \to \mathbb{R}$ be Lipschitz on geodesic ball $B_r(x)$ and $0 \notin \partial f(x)$. If S is defined as $S := \{y \in M : f(y) \leq f(x)\}$. Then one has*

$$\{v \in T_xM : f^\circ(x, v) \leq 0\} \subset T_S(x).$$

Proof. The claim can be reached combining the results contained in Remark 4.2 and [22, Theorem 2.4.7]. Along the same lines as [22, Theorem 2.4.7], it suffices to prove that any $v \in T_xM$ for which $f^\circ(x, v) < 0$ belongs to $T_S(x)$. Let $v \in T_xM$ such that $f^\circ(x, v) < 0$ and $(z_i) \subset \exp_x^{-1}(S \cap B_r(x))$ be any sequence converging to 0_x and $t_i \in (0, \infty)$ be any sequence decreasing to 0, then there is a number $\delta > 0$ such that for all i large enough,

$$f(\exp_x(z_i + t_i v)) \leq f(\exp_x(z_i)) - \delta t_i \leq f(x) - \delta t_i.$$

So for all i large enough, $z_i + t_i v \in \exp_x^{-1}(S \cap B_r(x))$ which means $v \in T_S(x)$, as required. □

The following corollary is a direct consequence of the previous theorem.

Corollary 4.12. *Let M be a complete finite dimensional Riemannian manifold (or else, an infinite dimensional manifold with the property that every two points of M are connected by a minimizing geodesic), $x \in M$, $f : M \to \mathbb{R}$ be Lipschitz on geodesic ball $B_r(x)$ and $0 \notin \partial f(x)$. If S is defined as $S := \{y \in M : f(y) \leq f(x)\}$. Then*

$$N_S(x) \subset \bigcup_{\lambda \geq 0} \lambda \partial f(x).$$

5. A characterization of epi-Lipschitz subsets of complete finite dimensional Riemannian manifolds

In this section, we characterize epi-Lipschitz subsets of complete finite dimensional Riemannian manifolds by using the results of the previous sections. This class of sets is of particular importance since it includes closed convex sets with nonempty interior and sets defined by finite smooth inequality constraints satisfying a nondegeneracy assumption. Throughout this section manifolds under consideration are assumed to be finite dimensional.

Definition 5.1. Let M be a Riemannian manifold and $S \subset M$. The set S is said to be epi-Lipschitz if at every point $x \in S$,

$$N_S(x) \cap (-N_S(x)) = \{0\}.$$

Note that the previous definition is equivalent to say that $\operatorname{int} T_S(x) \neq \emptyset$.

Let us define a subset of T_xM as follows:

$$H_S(x) = \{v \in T_xM : \exists \varepsilon > 0, (z + tv') \in \exp_x^{-1}(S \cap U), \forall (z, v', t) \in (B(0_x, \varepsilon) \cap \exp_x^{-1}(S \cap U)) \times B(v, \varepsilon) \times [0, \varepsilon)\},$$

where U is any normal neighborhood of x. Then the following theorem can be deduced from [13, Theorem 2].

Theorem 5.2. *Let M be a complete Riemannian manifold and S be a closed subset of M. Then S is epi-Lipschitz at $x \in \partial S$ if and only if*

$$\emptyset \neq \operatorname{int} T_S(x) = H_S(x).$$

Example 5.3. (a) Let $f : M \to \mathbb{R}$ be a Lipschitz function defined on a complete Riemannian manifold M. Then $\operatorname{epi} f := \{(x, r) \in M \times \mathbb{R} : f(x) \leq r\}$ is an epi-Lipschitz subset of manifold $M \times \mathbb{R}$. Indeed, if $f(x) < r$ then $(x, r) \in \operatorname{int} \operatorname{epi} f$ which implies $\operatorname{int} T_{\operatorname{epi} f}(x, r) \neq \emptyset$, and if $f(x) = r$ then $(0_x, 1) \in H_{\operatorname{epi} f}(x, f(x))$.

(b) Every closed convex set S with nonempty interior in a complete Riemannian manifold M is epi-Lipschitz, since for arbitrary $x \in S$, $\exp_x^{-1}(x') \in H_S(x)$, where $x' \in \operatorname{int} S$.

Remark 5.4. It can be verified that if S is a closed subset of $\mathbb{R}^n$ which is epi-Lipschitz at a boundary point x, then

$$H_S(x) = -H_{\mathbb{R}^n \setminus \operatorname{int} S}(x).$$

Also, if S is a closed subset of Riemannian manifold M which is epi-Lipschitz at a boundary point x and $\exp_x : V \to U$ is diffeomorphism, then one has

$$H_S(x) = H_{\exp_x^{-1}(U \cap S)}(0_x) = -H_{V \setminus \operatorname{int} \exp_x^{-1}(U \cap S)}(0_x).$$

Moreover,

$$H_{V \setminus \operatorname{int} \exp_x^{-1}(U \cap S)}(0_x) = H_{\exp_x^{-1}(U \cap (M \setminus \operatorname{int} S))}(0_x) = H_{M \setminus \operatorname{int} S}(x),$$

so that $H_S(x) = -H_{M \setminus \operatorname{int} S}(x)$.

It means that if M is a complete Riemannian manifold and S is a closed subset of M, and if S is epi-Lipschitz at $x \in \partial S$, then

$$\emptyset \neq \operatorname{int} T_S(x) = H_S(x) = -H_{M \setminus \operatorname{int} S}(x) = -\operatorname{int} T_{M \setminus \operatorname{int} S}(x).$$

Let us invoke another definition of generalized gradient of a locally Lipschitz function. The celebrated theorem of Rademacher [3, Theorem 5.7] asserts that every locally Lipschitz real-valued function f, is almost everywhere differentiable, so that Ω_f (the set on which f is differentiable) is dense in M. The differential of f defined on a Riemannian manifold M, can be used to generate its generalized gradient, as depicted in the following formula.

Lemma 5.5. *Let $f : M \to \mathbb{R}$ be locally Lipschitz on a Riemannian manifold M, then*

$$\partial f(x) = \overline{\operatorname{co}}\{\lim_{q \to \infty} \mathrm{d}f(x_q) : \{x_q\} \subseteq \Omega_f, x_q \to x\}.$$

It is worthwhile to mention that $\lim_{q \to \infty} \mathrm{d}f(x_q)$ in the previous lemma is obtained as follows. Let $\xi_i \in T_{x_i}M^*$, $i = 1, 2, \ldots$ be a sequence of cotangent vectors of M and let $\xi \in T_xM^*$. We say ξ_i converges to ξ, denoted by $\lim \xi_i = \xi$, provided that $x_i \to x$ and, for any C^∞ vector field X, $\langle \xi_i, X(x_i) \rangle \to \langle \xi, X(x) \rangle$. Let (U, φ)be a local chart neighborhood with $x \in U$. Since $x_i \to x$, we may assume without loss of generality that $x_i \in U$ for all i. Then $\lim \xi_i = \xi$ if and only if $\langle \xi_i, (\frac{\partial}{\partial x^n})_{x_i} \rangle \to \langle \xi, (\frac{\partial}{\partial x^n})_x \rangle$ for $n \leq \dim(M)$. The latter is clearly equivalent to $(\mathrm{d}\varphi^{-1})^*_{\varphi(x_i)}\xi_i \to (\mathrm{d}\varphi^{-1})^*_{\varphi(x)}\xi$.

Theorem 5.6. *Let M be a complete Riemannian manifold and S be a closed subset of M. Then the following assertions are equivalent.*

(a) *S is epi-Lipschitz.*
(b) *There is a locally Lipschitz function $f : M \to \mathbb{R}$ such that*
  (i) *$S = \{x \in M : f(x) \leq 0\}$,*
  (ii) *If $f(x) = 0$, then $0 \notin \partial f(x)$,*
  (iii) *$\partial S = \partial(\mathrm{int} S) = \{x \in M : f(x) = 0\}$.*

Proof. Let us prove the implication (b) ⇒ (a). Suppose that $f(x) < 0$. Then $\partial d_S(x) = \{0\}$ and Lemma 4.9 implies $N_S(x) = \{0\}$, which means that S is epi-Lipschitz at x. Now let $f(x) = 0$. We conclude from Corollary 4.12 that

$$N_S(x) \subset \bigcup_{\lambda \geq 0} \lambda \partial f(x).$$

Since $0 \notin \partial f(x)$, we have $N_S(x) \cap (-N_S(x)) = \{0\}$.

Now we are going to prove the converse implication. Note that if there is a locally Lipschitz function $f : M \to \mathbb{R}$ such that

(i) $S = \{x \in M : f(x) \leq 0\}$,
(ii) If $f(x) = 0$, then $0 \notin \partial f(x)$. Then, we can prove the third condition. Indeed it suffices to prove for every $x_0 \in M$ with $f(x_0) = 0$, $x_0 \in \overline{\mathrm{int} S} \setminus \mathrm{int} S$. By separation theorem, there exist $v_0 \in T_{x_0}M$ with $\|v_0\| = 1$ and $a > 0$ such that for all $x^* \in \partial f(x_0)$,

$$\langle x^*, v_0 \rangle > a.$$

By Theorem 2.10, for $a > 0$, there exists $\delta > 0$ with $\delta \leq r$ where $B_r(x_0)$ is a geodesic ball around x_0, such that if $d(x, x_0) < \delta$, then for $x^* \in \partial f(x)$, we have

$$\langle L_{xx_0}(x^*), v_0 \rangle > a.$$

Now, we define the Lipschitz function $\psi : (-\delta, \delta) \to \mathbb{R}$ by $\psi(t) = f \circ \exp_{x_0}(tv_0)$. By Proposition 3.4,

$$\partial \psi(t) \subseteq \{\langle \xi, d\exp_{x_0}(tv_0)(v_0) \rangle : \xi \in \partial f(\exp_{x_0}(tv_0))\}.$$

On the other hand, from Remark 2.8, for all t small enough

$$\|L_{x_0 \exp_{x_0}(tv_0)} - d\exp_{x_0}(tv_0)\| \leq \frac{a}{2k},$$

where k is the Lipschitz constant of f on $\overline{B_r(x_0)}$. Hence $\partial \psi(t) > a/2$, by Lebourg's Mean Value Theorem, for all $t \in (0, \delta)$,

$$\psi(t) \geq ta/2.$$

It means $f \circ \exp_{x_0}(tv_0) \geq ta/2 > 0$. Thus $\exp_{x_0}(tv_0) \notin S$ which implies $x_0 \notin \mathrm{int} S$. Moreover, if $t \in (-\delta, 0]$, then $\psi(t) \leq 0$ and $x_0 \in \overline{\mathrm{int} S}$.

Now, we prove the existence of the locally Lipschitz function f satisfying (i) and (ii). Define the locally Lipschitz function $\Delta_S : M \to \mathbb{R}$ by $\Delta_S(x) = d_S(x) - d_{M \setminus S}(x)$. Obviously,

$$S = \{x \in M : \Delta_S(x) \leq 0\}.$$

We are going to prove that $0 \notin \partial \Delta_S(\bar{x})$ for every $\bar{x}$ such that $\Delta_S(\bar{x}) = 0$. We claim that $\langle \xi, \bar{v} \rangle < 0$ for all $\xi \in \partial \Delta_S(\bar{x})$ and for all $0 \neq \bar{v} \in \mathrm{int} T_S(\bar{x})$. To prove the claim, it is sufficient to prove for all $0 \neq \bar{v} \in H_S(\bar{x})$, there is $\varepsilon > 0$ such that $\langle \mathrm{d}\Delta_S(x), d\exp_{\bar{x}}(\exp_{\bar{x}}^{-1}(x))(\bar{v}) \rangle \leq -\varepsilon$ for all $x \in B(\bar{x}, \varepsilon)$ provided that Δ_S is differentiable at x. Then, Theorem 5.2 and Lemma 5.5 imply the claim. Suppose $0 \neq \bar{v} \in \mathrm{int} T_S(\bar{x})$, one can prove that there exists $\varepsilon > 0$ such that for all $z \in B(0_{\bar{x}}, \varepsilon)$ and $v \in B(\bar{v}, \varepsilon)$ and $t \in [0, \varepsilon)$,

$$d_S(\exp_{\bar{x}}(z + tv)) \leq d_S(\exp_{\bar{x}}(z)). \tag{5.1}$$

By polarity there exists $\delta > 0$ such that $\langle \xi, \bar{v} \rangle \leq -\delta \|\xi\|$ for all $\xi \in N_S(\bar{x})$. If (5.1) fails to hold, there exist sequences $\{z_i\} \subset T_{\bar{x}}M$, $\{v_i\} \subset T_{\bar{x}}M$ and $\{t_i\} \subset \mathbb{R}$ converging to $0_{\bar{x}}, \bar{v}, 0$ such that

$$d_S(\exp_{\bar{x}}(z_i + t_i v_i)) > d_S(\exp_{\bar{x}}(z_i)).$$

Applying [10, Theorem 3.11], there exist t_{0_i}, y_i and $\xi_i \in \partial_P d_S(y_i)$ with $d(y_i, \exp_{\bar{x}}(z_i + t_{0_i} v_i)) < 1/i$ such that

$$0 < \langle \xi_i, L_{\exp_{\bar{x}}(z_i + t_{0_i} v_i) y_i}(d\exp_{\bar{x}}(z_i + t_{0_i} v_i)(v_i)) \rangle + 1/i.$$

Take a subsequence if necessary to have $L_{y_i\bar{x}}(\xi_i/\|\xi_i\|)$ converges to a limit ξ which by Lemma 4.8 is in $N_S(\bar{x})$. So Remark 2.8 implies

$$0 \leq \langle \xi, \bar{v} \rangle \leq -\delta \|\xi\| = -\delta,$$

which is a contradiction. Thus, there exists $\varepsilon > 0$ such that for all $z \in B(0_{\bar{x}}, \varepsilon)$ and $v \in B(\bar{v}, \varepsilon)$ and $t \in [0, \varepsilon)$, (5.1) holds.

On the other hand, there is $\epsilon \in (0, 1)$ such that for all $(z, v, t) \in (B(0_{\bar{x}}, \epsilon) \cap \exp_{\bar{x}}^{-1}(U \cap S)) \times B(\bar{v}, \epsilon) \times [0, \epsilon)$,

$$(z + tv) \in \exp_{\bar{x}}^{-1}(S \cap U),$$

where U is any normal neighborhood of x. Let $\varepsilon' = \min\{\epsilon, \varepsilon, r\}$ where $B_r(\bar{x})$ is a geodesic ball around $\bar{x}$, and let $\epsilon' = (\varepsilon' - \varepsilon'^2)/2$ and $x \in B(\bar{x}, \epsilon')$ such that Δ_M is differentiable at x which is in $M \setminus \operatorname{int} S$ (the proof is similar when $x \in \bar{S}$). It is easy to see that (5.1) implies for t small enough, $d_S(\exp_{\bar{x}}(\exp_{\bar{x}}^{-1}(x) - t\bar{v})) \geq d_S(x) + \varepsilon' t$. So that

$$\varepsilon' t \leq \Delta_S(\exp_{\bar{x}}(\exp_{\bar{x}}^{-1}(x) - t\bar{v})) - \Delta_S(x) = \langle \mathrm{d}\Delta_S(x), d\exp_{\bar{x}}(\exp_{\bar{x}}^{-1}(x))(-t\bar{v}) \rangle + o(t),$$

and we get

$$\langle \mathrm{d}\Delta_S(x), d\exp_{\bar{x}}(\exp_{\bar{x}}^{-1}(x))(\bar{v}) \rangle \leq -\varepsilon',$$

as required. □

Example 5.7. Let M be a complete Riemannian manifold, $f_1, \ldots, f_k : M \to \mathbb{R}$ be C^1 functions and assume that for all $x \in M$ the set $\{\mathrm{d}f_i(x) \in T_xM^*, i \in I(x)\}$ is independent in T_xM^*, where $I(x) = \{i \in \{1, \ldots, k\} : f_i(x) = 0\}$. Then $S := \{x \in M : f_i(x) \leq 0, i = 1, \ldots, k\}$ is an epi-Lipschitz subset of M. Indeed, if we define $f : M \to \mathbb{R}$ by $f(x) = \max_{i\in\{1,\ldots,k\}} f_i(x)$, then $\partial f(x) = \mathrm{co}\{\mathrm{d}f_i(x) : i \in I(x)\}$. Hence f satisfies the conditions of part (b) of Theorem 5.6.

Acknowledgements

The present research was done while the second author was on sabbatical at the Department of Mathematics, University of Toronto, whose hospitality is gratefully acknowledged. The authors are also indebted to B. S. Mordukhovich for his encouragement and helpful discussions on the subject of this paper.

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