A Proof of the Consistency of the Finite Difference Technique on Sparse Grids

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Abstract

In this paper, we give a proof of the consistency of the finite difference technique on regular sparse grids [7, 18]. We introduce an extrapolation-type discretization of differential operators on sparse grids based on the idea of the combination technique and we show the consistency of this discretization. The equivalence of the new method with that of [7, 18] is established.

Key words. finite differences, wavelets, interpolets, sparse grids, combination technique
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1 Introduction

Consider an univariate scaling function ϕ which induces a multi-resolution analysis $V^l \subset V^{l+1} \subset ...$ of $L_2(\mathbb{R})$, see [4], such that

$$\overline{\bigcup_{l \in \mathbb{N}} V^l} = L_2(\mathbb{R}) \ , \quad V^l = \mathrm{span}\{\phi(2^l x - s) \ \mid \ s \in \mathbb{Z}\} \ .$$

The wavelets $\psi_{(l,t)}$ are the basis functions of the complementary spaces W_l:

$$W_l \oplus V^{l-1} \ = \ V^l \ , \quad W_l = \mathrm{span}\{\psi_{(l,t)} \ \mid \ t \in \mathbb{Z}\} \ .$$

In this paper, we consider scaling functions and wavelets of the interpolet family which is introduced in section 2. A simple way to the multivariate case is to use (in general anisotropic) tensor product wavelets, i.e.

$$\psi_{(\mathbf{l},\mathbf{t})}(\mathbf{x}) := \psi_{(l_1,t_1)}(x_1) \cdot ... \cdot \psi_{(l_d,t_d)}(x_d) \tag{1}$$

where $\mathbf{x} = (x_1, .., x_d)$, $\mathbf{l} = (l_1, .., l_d)$, and $\mathbf{t} = (t_1, .., t_d)$. Besides its simplicity this approach has a remarkable advantage with respect to its approximation power. Under some mild assumptions on the 1D wavelets there hold estimates of the type

$$\inf_{u_h \in \oplus_{|\mathbf{l}|_1 \leq n} W_\mathbf{l}} \|u - u_h\|_0 \quad \leq \quad C 2^{-\alpha n} \| \frac{\partial^{d\alpha} u}{\partial x_1^\alpha \cdot .. \cdot \partial x_d^\alpha} \|_0 \quad \text{where} \qquad |\mathbf{l}|_1 := \sum_{i=1}^d l_i \ . \tag{2}$$

If u has a compact support or decays sufficiently fast, then only $O(2^n n^{d-1})$ wavelets in $\oplus_\mathbf{l} W_\mathbf{l}$ are necessary for the representation of u_h. In contrast to this, $O(2^{dn})$ degrees of freedom are required to achieve a similar accuracy if a single scale space $V^\mathbf{l} := V^{l_1} \otimes .. \otimes V^{l_d}$ is used for the approximation. Standard references on this topic are [6, 9, 21].

In the last years, some techniques were developed to exploit the power of sparse grids for the solution of PDEs. A quite simple approach is to use the wavelets of (1) as trial functions for a Galerkin discretization [1, 2, 23]. Another approach is to use the tensor product wavelets as some sort of trial functions and to discretize differential operators by means of finite differences (FD) [7, 12, 18]. For this approach, consistency and stability are not easy to prove. Consistency was considered in [18] for the special case of so-called regular sparse grids where the trial space is $\oplus_{\mathbf{l} \in \Lambda} W_\mathbf{l}$. Here, Λ is a given finite index set, for example $\Lambda = \{\mathbf{l} \mid |\mathbf{l}|_1 \leq n\}$. As far as we know, there exists no proof of the stability of the discretization for general index sets Λ. Nevertheless, there is strong numerical evidence for the stability, see [18]. The advantage of FD on sparse grids is the relatively simple evaluation of discrete differential operators, when compared to the Galerkin approach. Furthermore, the FD technique is more flexible, if special properties (e.g. mass conservation) have to be included into the discretization.

There is a third way to exploit the idea of sparse grids: The so-called combination technique [3, 8, 11, 14, 15, 16, 17] extrapolates a very accurate solution on $\oplus_{\mathbf{l} \in \Lambda} W_\mathbf{l}$ from solutions (FD or Galerkin) on subspaces $V^\mathbf{l}$, $\mathbf{l} \in \Lambda$.

In this paper, we show the equivalence of the FD discretization of differential operators on sparse grids and the extrapolation formula of the combination technique applied to the discretized differential operators on the rectangular subgrids which correspond to $V^\mathbf{l}$. We will use this result to give a new proof of the consistency of the sparse grid FD technique which improves previous results [18] in the following points:

- We need weaker smoothness requirements.
- The proof allows for quite general index sets Λ.
- Higher order discretizations and even compact FD schemes are covered.

This paper is organized as follows. In section 2 we briefly describe interpolet wavelets. We deal with the interpolation error associated to the interpolet approximation and the consistency error of general FD schemes. In section 3 an extrapolation-type discretization on sparse grids is introduced and estimates for the consistency error are given. In the last section we show that this extrapolation scheme is just another form of the FD method on sparse grids.

2 Prerequisites

The basic properties and the efficiency of our sparse grid FD technique essentially rely on the simplicity of the underlying 1D wavelets, i.e. of the interpolets [5]. For our purpose it is convenient to consider interpolets of order N on the interval $[0,1]$ without any boundary conditions, see [13]. In this case the scaling functions $\phi^{(l,s)}$ are defined for levels $l \geq l_0 := \lfloor log_2(N) \rfloor$ only. In the remainder of this paper we will denote by

$$\mathbf{\Phi}^l := \{\phi^{(l,s)}\}_{s \in S(l)} \ , \quad S(l) := \{0, ..., 2^l\}$$

the *row* vector of basis functions of V^l, i.e. $V^l :=$ span $\mathbf{\Phi}^l$. Basic properties of the interpolet scaling functions are:

$$\phi^{(l,s)}(x) = 0 \ \text{ if } x \notin]2^{-l}(s-2N), 2^{-l}(s+2N)[\quad \text{(compact support)} \ , \tag{3}$$

$$\|\phi^{(l,s)}\|_\infty = 1 \quad \text{(normalization)} \ , \tag{4}$$

$$\phi^{(l,s)}(t2^{-l}) = \delta(s-t) \quad \text{(interpolating)} \ , \tag{5}$$

$$\forall\ 0 \leq i < N,\ 0 \leq x \leq 1 :\ x^i = \sum_{s \in S(l)} s^i \phi^{(l,s)}(x) \quad \text{(polynomial exactness)} . \tag{6}$$

For $N = 2$, the scaling functions $\phi^{(l,s)}$ are just dilates/translates of the hat function $\phi^{(l,s)}(x) = \max(0, 1 - |2^l x - s|)$.

Another property of the interpolet family is the definition of the wavelets $\psi_{(l,t)}$ as the odd-numbered scaling functions $\phi^{(l,2t+1)}$. More precisely,

$$\psi_{(l,t)} := \begin{cases} \phi^{(l,2t+1)} & \text{if } l > l_0 \quad \text{and} \quad t \in T(l) := \{0, ..., 2^{l-1} - 1\} \\ \phi^{(l,t)} & \text{if } l = l_0 \quad \text{and} \quad t \in T(l_0) := S(l_0) \ . \end{cases} \tag{7}$$

In the remainder of this paper we use the following notations:

$\boldsymbol{\Psi}_l := \{\psi_{(l,t)}\}_{t\in T(l)}$	row vector of wavelets of level l ,
$\mathbf{u}^l := \{u^{(l,s)}\}_{s\in S(l)}$	column vector of scaling function coefficients ,
$\mathbf{u}_l := \{u_{(l,t)}\}_{t\in T(l)}$	column vector of wavelet coefficients ,
$\Omega^l := 2^{-l}S(l)$	uniform grid with mesh size 2^{-l} .

Subscript indices like in $u_{(l,t)}$ are used mainly for wavelets and wavelet coefficients, while superscript indices are used mainly for the scaling functions. According to the definition of $\mathbf{u}_l, \boldsymbol{\Psi}_l$ and $\mathbf{u}^l, \boldsymbol{\Phi}^l$, we define the scalar products

$$\boldsymbol{\Psi}_l \cdot \mathbf{u}_l := \sum_{t\in T(l)} u_{(l,t)} \cdot \psi_{(l,t)} \quad \text{and} \quad \boldsymbol{\Phi}^l \cdot \mathbf{u}^l := \sum_{s\in S(l)} u^{(l,s)} \cdot \phi^{(l,s)} \ .$$

The following univariate restriction, interpolation and finite difference operators are the building blocks of the extrapolation discretization as well as of the finite difference technique on sparse grids:

$$\begin{array}{llll} P^l:\ C([0,1]) & \to\ \mathbb{R}^{\#S(l)}\ , & P^l(u) := u(\Omega^l) & \text{(restriction)}\ , \\ I^l:\ \mathbb{R}^{\#S(l)} & \to\ C([0,1])\ , & I^l(\mathbf{u}^l) := \boldsymbol{\Phi}^l \cdot \mathbf{u}^l & \text{(interpolation)}\ , \\ D^l:\ \mathbb{R}^{\#S(l)} & \to\ \mathbb{R}^{\#S(l)}\ , & D^l(\mathbf{u}^l) := D^l \cdot \mathbf{u}^l & \text{(finite difference)}\ . \end{array}$$

Here, we consider only *linear* finite difference operators. These can be written as matrix vector products. This includes the usual explicit or compact FD schemes of arbitrary order, but excludes upwind or ENO schemes, see [19]. The matrix D^l is the matrix representation of $D^l(.)$. For the sake of simplicity we restrict our considerations to the discretization of the operator ∂_{xx} in this paper.

Let M be the order of consistency of $D^l(.)$. Then there holds for the matrix D^l:

$$\forall s \in S(l)\ :\quad \sum_{i:\, s+i\in S(l)} D^l_{s,s+i} \cdot i^r \ = \ \begin{cases} 4^l \cdot 2 & \text{if } r = 2\ , \\ 0 & r \in \{0, ..., M+1\}\backslash\{2\}\ . \end{cases} \tag{8}$$

Besides the linearity of $D^l(.)$ we require for estimates of the consistency error that the off-diagonal entries $D^l_{s,t}$, $s \neq t$, of D^l decay fast away from the diagonal. More precisely, there shall exist $K_D < \infty$ such that for all $l \geq l_0$ holds:

$$\max\left\{ \frac{1}{4^l} \sum_{i:\, s+i\in S(l)} |D^l_{s,s+i}||i|^k \ \middle|\ s \in S(l)\ ,\ 0 \leq k \leq M+1 \right\} \leq K_D\ .$$

Clearly, this is satisfied for the usual explicit schemes. For compact schemes things are not so obvious. However, for the periodic case we can consider each row of D^l, for example $\{D^l_{0,i}\}_{i\in S(l)}$, as the discrete Fourier transform of the arbitrarily smooth symbol of the compact FD scheme. Therefore, the entries $D^l_{0,i}$ decay sufficiently fast in this case.
Now, we summarize those properties of the univariate operators P^l, I^l and D^l which will be used in the consistency proof for the extrapolation-type discretization in section 3. The proof essentially relies on estimates for the interpolation error $u - I^l \circ P^l(u)$, the consistency error $P^l(\partial_{xx}u) - D^l \circ P^l(u)$ and the linearity of the corresponding error functionals with respect to the argument u. In the multivariate case this will allow to estimate terms like the error of interpolation in x_2-direction of the consistency error of a FD operator which acts in x_1-direction. However, in this schedule, assumptions on the smoothness of the function u are required. Here, we will use Hölder spaces for the smoothness characterization. As usual, we denote by Δ_h, $h \neq 0$, the difference operator $\Delta_h u(x) := u(x+h) - u(x)$. Furthermore, for $\alpha > 0$, we define

$$\{\alpha\}^+ := \alpha - \lfloor\alpha\rfloor_- \;, \quad \lfloor\alpha\rfloor_- := \begin{cases} \lfloor\alpha\rfloor & \text{if } \alpha \notin \mathbb{N} \\ \alpha - 1 & \text{if } \alpha \in \mathbb{N} \end{cases} .$$

Now, u is in the function space $\mathcal{C}^\alpha([0,1])$, $\alpha > 0$, if u is $\lfloor\alpha\rfloor_-$ times continuously differentiable and if the associated semi-norm

$$|u|_\alpha := sup\left\{ \frac{|\Delta_h u^{(\lfloor\alpha\rfloor_-)}(x)|}{|h|^{\{\alpha\}^+}} \;\middle|\; x, x+h \in [0,1] \right\}$$

is finite. Note that, for $\alpha \in \mathbb{N}$, $\mathcal{C}^\alpha$ is the space of functions whose $(\alpha-1)$th derivative is Lipschitz continuous, i.e. $\mathcal{C}^\alpha$ is larger than the space C^α of α-times continuously differentiable functions.

Proposition 2.1 (Interpolation and Error Functionals).

(i) *The interpolation operators $\{I^l\}_{l\geq l_0}$ are uniformly bounded:*

$$\|I^l(\mathbf{u}^l)\|_\infty \;\; \leq \;\; C\|\mathbf{u}^l\|_\infty \;. \tag{9}$$

Furthermore, for $l \geq l_0$, let $E_D(.,l) : C^2([0,1]) \to \mathbb{R}^{\#S(l)}$ be the functional of the consistency error in the nodes of the grid Ω^l:

$$E_D(u,l) := P^l(\partial_{xx}u) \;-\; D^l \circ P^l(u) \;\in \mathbb{R}^{\#S(l)} \;.$$

Denote by $E_I(.,l): C([0,1]) \to C([0,1])$ the interpolation error of the interpolet approximation:

$$E_I(u,l)(x) := u(x) - (I^l \circ P^l(u))(x) \; .$$

Then there holds:

(ii) *$E_D(.,l)$ and $E_I(.,l)$ are linear in the first argument.*

(iii) *If $u \in \mathcal{C}^\alpha$, $\alpha > 2$, then*

$$\|E_D(u,l)\|_\infty \quad \le \quad C 2^{-\min(M,\alpha-2)l} |u|_{\min(M+2,\alpha)} \; .$$

(iv) *If $u \in \mathcal{C}^\alpha$, $\alpha > 0$, then*

$$\|E_I(u,l)\|_\infty \quad \le \quad C 2^{-\min(N,\alpha)l} |u|_{\min(N,\alpha)} \; .$$

Proof. **(i)** follows from (3) and (4). Let $x \in [0,1]$, then

$$|I^l(\mathbf{u}^l)(x)| \quad = \quad \left| \sum_{s:\ \phi^{(l,s)}(x) \neq 0} u^{(l,s)} \cdot \phi^{(l,s)}(x) \right| \quad \overset{(3)}{\le} \quad 4N \|\mathbf{u}^l\|_\infty \; ,$$

since there are at most $4N$ indices s such that $\phi^{(l,s)}(x) \neq 0$.
(ii) trivial.
(iii) The proof of **(iii)** uses standard arguments. Let $h := 2^{-l}$, $x_s := s \cdot h$, and $k := \min(\lfloor \alpha \rfloor_- - 1, M)$. Let $E_D(u,l)_s$, $s \in S(l)$, be the sth component of $E_D(u,l)$. Taylor expansion for $u(x_{s+i})$ yields: $\quad E_D(u,l)_s =$

$$\begin{aligned} &= \quad u^{(2)}(x_s) - \sum_{i:\, s+i \in S(l)} D^l_{s,s+i} \left(\sum_{r=0}^{k} \frac{u^{(r)}(x_s)}{r!} i^r h^r + \frac{1}{k!} \int_{x_s}^{x_{s+i}} (x_{s+i} - t)^k u^{(k+1)}(t) \; dt \right) \\ &\overset{(8)}{=} \quad \frac{1}{k!} \sum_{i:\, s+i \in S(l)} D^l_{s,s+i} \left(\int_{x_s}^{x_{s+i}} (x_{s+i} - t)^k (u^{(k+1)}(t) - u^{(k+1)}(x_s)) \; dt \right) \; . \end{aligned}$$

Now, it is straightforward to use the Hölder continuity of $u^{(k+1)}$ to obtain assertion **(iii)**.
(iv) Let $k := \min(\lfloor \alpha \rfloor_-, N) - 1$ and $x \in [0,1]$. If $x = x_s$ for some $s \in S(l)$,

then $E_I(u,l)(x) = 0$, see (5). Otherwise, there is $s \in S(l)$ such that $x \in]x_s, x_{s+1}[$. Taylor expansion around x_s yields: $E_I(u,l)(x) =$

$$= \sum_{r=0}^{k} \frac{u^{(r)}(x_s)}{r!} \Big[(x - x_s)^r - \sum_{i:\, s+i \in S(l)} (x_{s+i} - x_s)^r \ \phi^{(l,s+i)}(x) \Big]$$
$$+ \frac{1}{k!} \int_{x_s}^{x} (x-t)^k u^{(k+1)}(t)dt - \frac{1}{k!} \sum_{i:\, s+i \in S(l)} \int_{x_s}^{x_{s+i}} (x_{s+i} - t)^k u^{(k+1)}(t)dt \ \phi^{(l,s+i)}(x) \ .$$

The term in the brackets [...] is zero, see (6). Now, the assertion follows from the same arguments as in the proof of **(iii)**. $\square$

3 Consistency

In this section, we introduce and analyse the extrapolation-type discretization on sparse grids. To this end, first we describe the notation for the multivariate case and recall the combination technique [8, 11, 16].

For our approach, the multivariate wavelets are tensor products of univariate interpolets, see (1). This suggests the following settings:

$V^{\mathbf{l}}$	$:=$	$V^{l_1} \otimes ... \otimes V^{l_d}$,	(approximation space)
$W_{\mathbf{l}}$	$:=$	$W_{l_1} \otimes ... \otimes W_{l_d}$,	(wavelet space)
$S(\mathbf{l})$	$:=$	$S(l_1) \otimes ... \otimes S(l_d)$,	(index set for scaling func.)
$T(\mathbf{l})$	$:=$	$T(l_1) \otimes ... \otimes T(l_d)$,	(index set for wavelets)
$\Omega^{\mathbf{l}}$	$:=$	$\Omega^{l_1} \otimes ... \otimes \Omega^{l_d}$,	(multivariate grid)
$P^{\mathbf{l}}(u)$	$:=$	$u(\Omega^{\mathbf{l}})$,	(restriction operator)
$\phi^{(\mathbf{l},\mathbf{s})}(\mathbf{x})$	$:=$	$\phi^{(l_1,s_1)}(x_1) \cdot ... \cdot \phi^{(l_d,s_d)}(x_d)$,	
$\mathbf{\Phi}^{\mathbf{l}}$	$:=$	$\{\phi^{(\mathbf{l},\mathbf{s})}\}_{\mathbf{s}\in S(\mathbf{l})}$,	(row vector)
$\mathbf{\Psi}_{\mathbf{l}}$	$:=$	$\{\psi_{(\mathbf{l},\mathbf{t})}\}_{\mathbf{t}\in T(\mathbf{l})}$,	(row vector)
$\mathbf{u}^{\mathbf{l}}$	$:=$	$\{u^{(\mathbf{l},\mathbf{s})}\}_{\mathbf{s}\in S(\mathbf{l})}$,	(column vector)
$\mathbf{u}_{\mathbf{l}}$	$:=$	$\{u_{(\mathbf{l},\mathbf{t})}\}_{\mathbf{t}\in T(\mathbf{l})}$,	(column vector)
$I^{\mathbf{l}}(\mathbf{u}^{\mathbf{l}})$	$:=$	$\mathbf{\Phi}^{\mathbf{l}} \cdot \mathbf{u}^{\mathbf{l}}$,	(interpolation operator)
$\Delta_{i,h} u(\mathbf{x})$	$:=$	$u(\mathbf{x} + h\mathbf{e}_i) - u(\mathbf{x})$,	(difference in ith coord. d.)
$\mathbf{l} \geq \mathbf{k}$	$:\Leftrightarrow$	$\forall\, 1 \leq i \leq d:\ l_i \geq k_i$	
$\mathbf{l} > \mathbf{k}$	$:\Leftrightarrow$	$\forall\, 1 \leq i \leq d:\ l_i > k_i$.	

Similar to the previous section, the scalar products $\mathbf{\Phi}^{\mathbf{l}} \cdot \mathbf{u}^{\mathbf{l}}$, $\mathbf{\Psi}_{\mathbf{l}} \cdot \mathbf{u}_{\mathbf{l}}$ denote the sums $\sum_{\mathbf{s}\in S(\mathbf{l})} u^{(\mathbf{l},\mathbf{s})} \cdot \phi^{(\mathbf{l},\mathbf{s})}$ and $\sum_{\mathbf{t}\in T(\mathbf{l})} u_{(\mathbf{l},\mathbf{t})} \cdot \psi_{(\mathbf{l},\mathbf{t})}$, respectively.

The combination technique calculates a numerical solution $u^\Lambda \in V^\Lambda := \oplus_{\mathbf{l}\in\Lambda} W_\mathbf{l}$ of a given PDE by means of extrapolation from solutions $u^\mathbf{l} \in V^\mathbf{l}$, $\mathbf{l} \in \Lambda$, of the given PDE:

$$u^\Lambda := \sum_{\mathbf{l}\in\Lambda} f(\mathbf{l}) \cdot u^\mathbf{l} \ . \tag{10}$$

The weights $f(\mathbf{l})$ are only dependent on $\mathbf{l}$ and on the finite index set Λ. Their definition and some basic properties are given later in this section. The extrapolation (10) leads to very accurate results provided the individual errors $u - u^\mathbf{l}$ can be written as

$$u - u^\mathbf{l} = E(u, \mathbf{l}) \ + \ \sum_{i=1}^{d} E_i(u, l_i) \ , \tag{11}$$

where u is the exact solution of the PDE and $E_i(u, l_i)$, $E(u, \mathbf{l})$ are functions ($[0,1]^d \to \mathbb{R}$) which decay with a certain rate α:

$$\|E_i(u, l_i)\|_\infty < C 2^{-\alpha l_i} \ , \quad \|E(u, \mathbf{l})\|_\infty < C 2^{-\alpha |\mathbf{l}|_1} \ .$$

Of course, C and α depend on u and the numerical scheme which was used for the calculation of $u^\mathbf{l}$. Although the single errors $u - u^\mathbf{l}$ may be of $O(1)$, the final error $u - u^\Lambda$ is quite small because of cancellation effects in (10). Now, the idea of the extrapolation-type discretization of ∂_{xx} on sparse grids is to exploit expansions of the error $\partial_{xx} u - I^\mathbf{l} \circ \left(D^{l_1} \otimes id\right) \circ P^\mathbf{l}(u)$ which are similar to those of (11). Therefore, we define the extrapolation discretization of ∂_{xx} as

$$D^\Lambda(u) \ := \ \sum_{\mathbf{l}\in\Lambda} f(\mathbf{l}) \cdot I^\mathbf{l} \circ (D^{l_1} \otimes id) \circ P^\mathbf{l}(u) \ . \tag{12}$$

The extrapolation weights $f(\mathbf{l})$ are recursively defined by :

$$f(\mathbf{l}) \ := \ \begin{cases} 0 & : \ \mathbf{l} \notin \Lambda \ , \\ 1 - \sum_{\mathbf{k}\geq\mathbf{l}, \mathbf{k}\neq\mathbf{l}} f(\mathbf{k}) & : \ \mathbf{l} \in \Lambda \ . \end{cases} \tag{13}$$

Other, more explicit characterizations are given e.g. in [15, 16]. Figure 1 shows three 2D examples for Λ and the corresponding weights $f(\mathbf{l})$. The left example is a so-called regular sparse grid: $\Lambda = \{\mathbf{l} \ | \ |\mathbf{l}|_1 \leq n\}$ where $l_0 = 1$ and $n = 5$.

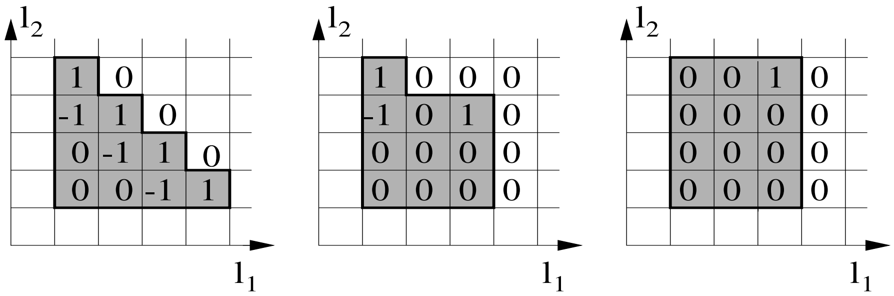

Figure 1: Λ and $f(\mathbf{l})$, $\mathbf{l} \in \Lambda$.

In the remainder of this paper we assume a minimal regularity condition for the index set Λ:

$$\mathbf{l} \in \Lambda \text{ and } \mathbf{k} \text{ such that } \ (l_0, ..., l_0) \leq \mathbf{k} \leq \mathbf{l} \quad \Rightarrow \quad \mathbf{k} \in \Lambda \ . \tag{14}$$

This means there are no holes allowed in Λ. In this case, the weights $f(.)$ possess a number of properties which will be used in the proof of Theorem (3.2) and in section 4:

Proposition 3.1 (Extrapolation Weights).

(i) *Let* $u = \sum_{\mathbf{l} \in \Lambda} u_{\mathbf{l}}$, $u_{\mathbf{l}} \in W_{\mathbf{l}}$. *Then*

$$u \ = \ \sum_{\mathbf{l} \in \Lambda} f(\mathbf{l}) \Big(\sum_{\mathbf{k} \leq \mathbf{l}} u_{\mathbf{k}} \Big) \ . \tag{15}$$

(ii) *For* $1 \leq i \leq d$, *and* $l \geq l_0$, *let* $\Lambda(i,l) := \{\mathbf{l} \in \Lambda \ \mid \ l_i = l\}$ *be a slice of* Λ. *Furthermore, let* $lmax(i) := \max\{l_i \ \mid \ \mathbf{l} \in \Lambda\}$ *be the finest level in the ith coordinate direction. Then*

$$\sum_{\mathbf{l} \in \Lambda(i,l)} f(\mathbf{l}) \ = \ \begin{cases} 1 & \text{if } l = lmax(i) \ , \\ 0 & \text{else} \ . \end{cases}$$

A few remarks are in order. **(i)** is an alternative *definition* of $f(.)$. One can show that the weights $f(\mathbf{l})$ are uniquely determined by property **(i)**. Assertion **(ii)** leads to the above-mentioned cancellation of large errors in (10). We will use this argument for the proof of Theorem 3.2 later on.

Proof. (Proposition 3.1) **(i)**

$$\sum_{\mathbf{l}\in\Lambda} f(\mathbf{l})\Big(\sum_{\mathbf{k}\leq\mathbf{l}} u_{\mathbf{k}}\Big) \;=\; \sum_{\mathbf{k}\in\Lambda} u_{\mathbf{k}}\Big(\sum_{\mathbf{l}\geq\mathbf{k}} f(\mathbf{l})\Big) \;=\; \sum_{\mathbf{k}\in\Lambda} u_{\mathbf{k}}\cdot 1$$

(ii) For simplicity of notation we consider only $i = 1$. Because of (14) there holds $\mathbf{l} := (lmax(1), l_0, ..., l_0) \in \Lambda$. Thus,

$$\sum_{\mathbf{k}\in\Lambda(1,lmax(1))} f(\mathbf{k}) \;=\; \sum_{\mathbf{k}\geq\mathbf{l}} f(\mathbf{k}) \;=\; 1\ .$$

The cases $l_0 \leq l < lmax(1)$ are proved by induction on l. Again because of (14), $\mathbf{l} := (l, l_0, ..., l_0) \in \Lambda$. Therefore,

$$1 = \sum_{\mathbf{k}\geq\mathbf{l}} f(\mathbf{k}) \;=\; \sum_{\mathbf{k}\in\Lambda(1,l)} f(\mathbf{k}) \;+\; \sum_{\mathbf{k}\in\Lambda(1,l+1)} f(\mathbf{k}) \;+\; ... \;+\; \sum_{\mathbf{k}\in\Lambda(1,lmax(1))} f(\mathbf{k})\ .$$

The assumption holds for $l+1$,...,$lmax(1)$. Thus, $\sum_{\mathbf{k}\in\Lambda(1,l)} f(\mathbf{k}) = 0$. □

Now, we consider the consistency of D^Λ. In order to simplify the presentation we treat only the 2D case. Nevertheless, all results hold in the general multivariate case as well. First, Hölder spaces are generalized to the multivariate case in a way which fits to the setting of *bounded mixed derivatives*. This leads to tensor products of univariate Hölder spaces. For $\boldsymbol{\alpha} := (\alpha_1, .., \alpha_d) > \mathbf{0}$, a function u is in $\mathcal{C}^{\boldsymbol{\alpha}}([0,1]^d)$ if the mixed derivative

$$v := \frac{\partial^{|\lfloor\boldsymbol{\alpha}\rfloor_-|_1} u}{\partial_{x_1}^{\lfloor\alpha_1\rfloor_-}..\partial_{x_d}^{\lfloor\alpha_d\rfloor_-}}$$

is continuous and if the associated semi-norm

$$|u|_{\boldsymbol{\alpha}} := \sup\left\{\frac{|\Delta_{1,h_1}\circ..\circ\Delta_{d,h_d}v|}{|h_1|^{\{\alpha_1\}^+}\cdot..\cdot|h_d|^{\{\alpha_d\}^+}}\;\;|\;\;\mathbf{x},\mathbf{x}+\mathbf{h}\in[0,1]^d, \mathbf{h}\neq\mathbf{0}\right\}$$

is finite. This semi-norm will play essentially the same role as $|.|_\alpha$ in Proposition 2.1.

In [18] the consistency error of the finite difference method was considered in the nodes of the joint grid $\Omega^\Lambda := \bigcup_{\mathbf{l}\in\Lambda}\Omega^{\mathbf{l}}$ only. In our case this leads to some simplifications and, hence, we follow [18] in this regard. Nevertheless, it is straightforward for our approach to obtain also global estimates on

$[0,1]^d$ (see the remarks below). Another advantage of our approach is its flexibility: very general finite difference schemes and interpolets are covered, while only the usual central schemes and the hat function were considered in [18]. Furthermore, a quite weak smoothness of u is required in the following Theorem ($u \in \mathcal{C}^{(\alpha,\beta)}), \alpha > 2, \beta > 0$), while $\frac{\partial^6 u}{\partial x^4 \partial y^2} \in C([0,1]^2)$ is needed in [18].

Theorem 3.2 (Consistency). *Let $P^\Lambda(u) := u(\Omega^\Lambda)$ be the restriction operator for the sparse grid Ω^Λ. Then there holds*

$$\begin{aligned} P^\Lambda\big(\partial_{xx}u\big) - P^\Lambda \circ D^\Lambda(u) &= -P^\Lambda\Big(\sum_{\mathbf{l}\in\Lambda} f(\mathbf{l})(I^{l_1}\otimes id)\circ E_I^y\big(E_D^x(u,l_1),l_2\big)\Big) \\ &\quad + P^\Lambda\Big(\big(I^{lmax(1)}\otimes id\big)\circ E_D^x(u,lmax(1))\Big). \quad (16) \end{aligned}$$

Now, if $u \in \mathcal{C}^{(\alpha,\beta)}([0,1]^2)$, $\alpha > 2$, $\beta > 0$, and Λ corresponds to a sparse grid, $\Lambda = \{\mathbf{l} \mid |\mathbf{l}|_1 \le n\}$, then we have

$$\begin{aligned} \|P^\Lambda\big(\partial_{xx}u\big) - P^\Lambda \circ D^\Lambda(u)\|_\infty &\le C|u|_{(\min(\alpha,M+2),\min(\beta,N))} n 2^{-min(\alpha-2,\beta,M,N)n} \\ &\quad + C\sup_{0\le y\le 1}\big\{|u(.,y)|_{\min(\alpha,M+2)}\big\} 2^{-\min(\alpha-2,M)n}. (17) \end{aligned}$$

In the above assertion E_D^x and E_I^y denote the 1D functionals E_D and E_I which act only with respect to the x or y coordinate.

Remark 3.1. **(i)** Estimate (17) expresses the huge advantage of the discretization D^Λ in comparison to conventional finite difference schemes. Consider for example $u \in C^4([0,1]^2)$. Then the usual finite difference schemes on a grid with mesh size 2^{-n} and 4^n nodes yield a consistency error of $O(2^{-2n})$. D^Λ yields almost the same accuracy $O(n2^{-2n})$, but only $O(n2^n)$ degrees of freedom are needed.
(ii) To obtain global estimates on $[0,1]^2$ one has to replace (16) by

$$\begin{aligned} \sum_{\mathbf{l}\in\Lambda} f(\mathbf{l}) I^{\mathbf{l}} \circ P^{\mathbf{l}}(\partial_{xx}u) \quad - \quad D^\Lambda(u) &= \big(I^{lmax(1)}\otimes id\big)\circ E_D^x(u,lmax(1)) \\ &\quad -\sum_{\mathbf{l}\in\Lambda} f(\mathbf{l})(I^{l_1}\otimes id)\circ E_I^y\big(E_D^x(u,l_1),l_2\big). \end{aligned}$$

Supremum-norm estimates of this term are actually used in the proof of Theorem 3.2. However, for global estimates of $\partial_{xx}u - D^\Lambda(u)$ one has to consider also the additional *interpolation error*

$$\partial_{xx} u - \sum_{\mathbf{l}\in\Lambda} f(\mathbf{l}) I^{\mathbf{l}} \circ P^{\mathbf{l}}(\partial_{xx} u) \ . \tag{18}$$

To this end, one can use the techniques developed in this paper, for example take $D^{l_1} = id$ in (12). Estimates of (18) in terms of Sobolev norms were given in [20].

(iii) Relation (16) is a very general result. The main prerequisite of (16) is the linearity of $E_D(.,l)$. Therefore, (16, 17) hold for arbitrary 'univariate', linear differential operators $a(x)\frac{\partial^m u(x,y)}{\partial x^m}$ with linear discretizations. Such operators appear in the discretization of e.g. the Laplace equation on stretched, cartesian grids, see Figure 2. However, there are discretizations which are

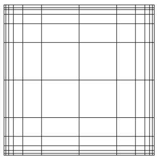

Figure 2: Stretched, cartesian grid.

not covered by Theorem 3.2 due to the lack of linearity. For example u-dependent upwind schemes, limiters or ENO schemes for convection-diffusion equations.

(iv) The error estimate for regular sparse grids suggests that one should chose $M = N$, i.e. the order of the interpolets and the order of the FD stencil should agree. Otherwise higher order FD stencils/interpolets do not pay off. In this sense, the collocation scheme introduced in [13] which –on regular sparse grids– can be considered as a special FD method, is not optimal, since there the 1D collocation scheme is of lower order than the interpolets.

Proof. (of Theorem 3.2) Consider an arbitrary function $w = \sum_{\mathbf{l}} w_{\mathbf{l}}\ ,\ w_{\mathbf{l}} \in W_{\mathbf{l}}$. There holds:

$$\begin{aligned} P^{\Lambda}(w) &= P^{\Lambda}\Big(\sum_{\mathbf{l}\in\Lambda} w_{\mathbf{l}} \ + \ \sum_{\mathbf{l}\notin\Lambda} w_{\mathbf{l}}\Big) \\ &= P^{\Lambda}\Big(\sum_{\mathbf{l}\in\Lambda} w_{\mathbf{l}}\Big) \ + \ 0 \qquad \text{since } \Psi_{l_i}(\Omega^{k_i}) = 0 \text{ for } k_i < l_i, (5),(7) \end{aligned}$$

$$
\begin{aligned}
&\overset{(15)}{=} \quad P^\Lambda\Big(\sum_{\mathbf{l}\in\Lambda} f(\mathbf{l})\cdot w^{\mathbf{l}}\Big) \qquad\qquad \text{where } w^{\mathbf{l}} := \sum_{\mathbf{k}\le\mathbf{l}} w_{\mathbf{k}}\\
&= \quad P^\Lambda\Big(\sum_{\mathbf{l}\in\Lambda} f(\mathbf{l})\cdot I^{\mathbf{l}}\circ P^{\mathbf{l}}\big(w^{\mathbf{l}}\big)\Big)\\
&= \quad P^\Lambda\Big(\sum_{\mathbf{l}\in\Lambda} f(\mathbf{l})\cdot I^{\mathbf{l}}\circ P^{\mathbf{l}}\big(w\big)\Big)\ .
\end{aligned}
$$

This relation for $w=\partial_{xx}u$ and (12) yield

$$
P^\Lambda\big(\partial_{xx}u\big) - P^\Lambda\circ D^\Lambda(u) = P^\Lambda\Big(\sum_{\mathbf{l}\in\Lambda} f(\mathbf{l})\cdot I^{\mathbf{l}}\circ\big(P^{\mathbf{l}}\big(\partial_{xx}u\big) - (D^{l_1}\otimes id)\circ P^{\mathbf{l}}u\big)\Big)\ .
$$

The term in the inner brackets reads

$$
P^{\mathbf{l}}\big(\partial_{xx}u\big) - (D^{l_1}\otimes id)\circ P^{\mathbf{l}}(u) = E^x_D(u(.,\Omega^{l_2}),l_1)\ .
$$

Therefore,

$$
\begin{aligned}
I^{\mathbf{l}}\circ\Big(P^{\mathbf{l}}\big(\partial_{xx}u\big) - (D^{l_1}\otimes id)\circ P^{\mathbf{l}}(u)\Big) \quad &= \quad (I^{l_1}\otimes id)\circ E^x_D(u(.,y),l_1)\\
&\quad -(I^{l_1}\otimes id)\circ E^y_I\big(E^x_D(u,l_1),l_2\big)\\
&=: \quad F(l_1) \quad - \quad G(\mathbf{l})\ .
\end{aligned}
$$

Now, the first assertion (16) follows from Proposition 3.1 **(ii)**:

$$
\sum_{\mathbf{l}\in\Lambda} f(\mathbf{l})F(l_1) \ = \ \sum_{l_1=l_0}^{lmax(1)} F(l_1)\Big(\sum_{\mathbf{l}\in\Lambda(1,l_1)} f(\mathbf{l})\Big) \ = \ F(lmax(1))\ .
$$

It remains to give estimates for $F(l_1)$ and $G(\mathbf{l})$. Combining Proposition 2.1 **(ii)** and (9) yields

$$
\|F(l_1)\|_\infty \quad\le\quad C2^{-\min(\alpha-2,M)l_1}\sup_{0\le y\le 1}|u(.,y)|_{\min(\alpha,M+2)}\ .
$$

For the second term, $G(\mathbf{l})$, we consider the functions

$$
g_s:\ y\ \mapsto\ E^x_D(u(.,y),l_1)_s\ ,\quad s\in S(l_1)\ .
$$

These functions are in $\mathcal{C}^\beta$ and there holds

$$
\begin{aligned}
|g_s|_\beta \quad &= \quad \sup_{y,h}\Big\{|E^x_D(v(.,y+h)-v(.,y),l_1)_s||h|^{-\{\beta\}^+}\Big\} \quad \text{where } v := \frac{\partial^{\lfloor\beta\rfloor^-}u}{\partial y^{\lfloor\beta\rfloor^-}}\\
&\le \quad C\sup_{y,h}\Big\{|v(.,y+h)-v(.,y)|_{\min(\alpha-2,M)}|h|^{-\{\beta\}^+}\Big\}2^{-\min(\alpha-2,M)l_1}\\
&\le \quad C|u|_{(\min(\alpha-2,M),\beta)}2^{-\min(\alpha-2,M)l_1}\ .
\end{aligned}
$$

Therefore,

$$\|E_I^y(g_s(y), l_2)\|_\infty \;\leq\; C|u|_{(\min(\alpha-2,M),\min(\beta,N))} 2^{-\min(\alpha-2,M)l_1} 2^{-\min(\beta,N)l_2} \;.$$

This finishes the proof of Theorem 3.2. □

4 Equivalence

In this section, we show that the extrapolation-type operator D^Λ (12) coincides with the sparse grid FD operator of [7, 18]. The corresponding algorithm for the evaluation consists of the following steps:

algorithm (FD)
Inverse interpolet transform of $\{u_{(\mathbf{l},\mathbf{t})}\}$ *only* with respect to the x_1-direction.
Application of the FD stencil in x_1-direction.
Interpolet transform in x_1-direction.

Now, we give a more precise formulation of this algorithm. To this end, the following notation is used: $\hat{\mathbf{l}}$ and $\hat{\mathbf{t}}$ are subvectors $(l_2, ..., l_d)$ and $(t_2, ..., t_d)$ of $\mathbf{l}$ and $\mathbf{t}$, respectively. The finest active level of a particular column $\{u_{(\mathbf{l},\mathbf{t})}\}_{(\hat{\mathbf{l}},\hat{\mathbf{t}})=const}$ of $\{\mathbf{u}_\mathbf{l}\}_{\mathbf{l}\in\Lambda}$ is denoted by

$$L(\hat{\mathbf{l}}) := \max\{l_1 \mid \mathbf{l} \in \Lambda\} \;.$$

Furthermore, let T_l be the matrix representation of the univariate interpolet transform $V^l \to \oplus_{k=l_0}^{l} W_k$. T_l is defined by $\mathbf{\Phi}^l =: \left(\mathbf{\Psi}_{l_0}, ..., \mathbf{\Psi}_l\right) \cdot T_l$. Note that T_l^{-1} is the matrix representation of the operator $P^l(.)$ with respect to the wavelet basis, i.e.

$$P^l\Big(\sum_{k=l_0}^{l} \mathbf{\Psi}_k \cdot \mathbf{u}_k\Big) = T_l^{-1} \cdot \{\mathbf{u}_k\}_{k=l_0}^{l} \;, \tag{19}$$

since $\mathbf{u}^l := T_l^{-1} \cdot \{\mathbf{u}_k\}_{k=l_0}^{l}$ is the vector of nodal values of $\mathbf{\Phi}^l \cdot \mathbf{u}^l$ on Ω^l, see (5). With these prerequisites, the result of algorithm (**FD**) reads

$$\sum_{\mathbf{l}\in\Lambda} \mathbf{\Psi}_\mathbf{l} \cdot \mathbf{v}_\mathbf{l}^{L(\hat{\mathbf{l}})} \tag{20}$$

where $\mathbf{v}_{\mathbf{l}}^{L(\hat{\mathbf{l}})}$ is defined column by column by

$$\{v_{(\mathbf{l},\mathbf{t})}^{L(\hat{\mathbf{l}})}\}_{(\hat{\mathbf{l}},\hat{\mathbf{t}})=const} := T_{L(\hat{\mathbf{l}})} \cdot D^{L(\hat{\mathbf{l}})} \cdot T_{L(\hat{\mathbf{l}})}^{-1} \cdot \{u_{(\mathbf{l},\mathbf{t})}\}_{(\hat{\mathbf{l}},\hat{\mathbf{t}})=const} \ .$$

Here, the superscript $L(\hat{\mathbf{l}})$ marks the dependency of $v_{(\mathbf{l},\mathbf{t})}^{L(\hat{\mathbf{l}})}$ on the operator $T_{L(\hat{\mathbf{l}})} \cdot D^{L(\hat{\mathbf{l}})} \cdot T_{L(\hat{\mathbf{l}})}^{-1}$. Later, in equation (21), we will deal with several coefficients $v_{(\mathbf{k},\mathbf{t})}^{l_1}$ which have the same subscript index $(\mathbf{k},\mathbf{t})$, but different superscript index l_1.
Now, consider the extrapolation-type discretization applied to $\sum_{\mathbf{k}\in\Lambda} \mathbf{\Psi}_{\mathbf{k}} \cdot \mathbf{u}_{\mathbf{k}}$:

$$\begin{aligned} D^{\Lambda}\big(\sum_{\mathbf{k}\in\Lambda} \mathbf{\Psi}_{\mathbf{k}} \cdot \mathbf{u}_{\mathbf{k}}\big) &= \sum_{\mathbf{l}\in\Lambda} f(\mathbf{l}) I^{\mathbf{l}} \circ (D^{l_1} \otimes id) \circ P^{\mathbf{l}}\big(\sum_{\mathbf{k}\in\Lambda} \mathbf{\Psi}_{\mathbf{k}} \cdot \mathbf{u}_{\mathbf{k}}\big) \\ &= \sum_{\mathbf{l}\in\Lambda} f(\mathbf{l}) I^{\mathbf{l}} \circ (D^{l_1} \otimes id) \circ P^{\mathbf{l}}\big(\sum_{\mathbf{k}\leq\mathbf{l}} \mathbf{\Psi}_{\mathbf{k}} \cdot \mathbf{u}_{\mathbf{k}}\big) \\ &= \sum_{\mathbf{l}\in\Lambda} f(\mathbf{l}) (I^{l_1} \circ D^{l_1} \circ P^{l_1}) \otimes id\big(\sum_{\mathbf{k}\leq\mathbf{l}} \mathbf{\Psi}_{\mathbf{k}} \cdot \mathbf{u}_{\mathbf{k}}\big) \\ &= \sum_{\mathbf{l}\in\Lambda} f(\mathbf{l}) \sum_{\mathbf{k}\leq\mathbf{l}} \mathbf{\Psi}_{\mathbf{k}} \cdot \mathbf{v}_{\mathbf{k}}^{l_1} \ . \end{aligned} \tag{21}$$

Here, we used $P^l(\mathbf{\Psi}_k) = \mathbf{\Psi}_k(\Omega^l) = \mathbf{0}$ for $k > l$ and the definition of $\mathbf{v}^l$. Thus, the equivalence of the finite difference algorithm **(FD)** and D^{Λ} follows from

$$\sum_{\mathbf{l}\in\Lambda} \mathbf{\Psi}_{\mathbf{l}} \cdot \mathbf{v}_{\mathbf{l}}^{L(\hat{\mathbf{l}})} \stackrel{!}{=} \sum_{\mathbf{l}\in\Lambda} f(\mathbf{l}) \sum_{\mathbf{k}\leq\mathbf{l}} \mathbf{\Psi}_{\mathbf{k}} \cdot \mathbf{v}_{\mathbf{k}}^{l_1} \ . \tag{22}$$

As an example consider the simple case $\Lambda = \{(l_0,l_0),(l_0+1,l_0),(l_0,l_0+1)\}$ which is the prototype for regular sparse grids in 2D, see Figure 3.

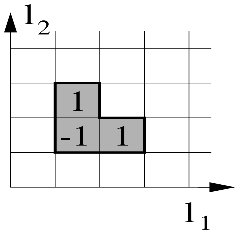

Figure 3: Λ for prototype sparse grid.

In this case the right-hand side of (22) reads:

$$\begin{array}{r} 1\cdot\left[\mathbf{\Psi}_{(l_0+1,l_0)}\cdot\mathbf{v}^{l_0+1}_{(l_0+1,l_0)}+\mathbf{\Psi}_{(l_0,l_0)}\cdot\mathbf{v}^{l_0+1}_{(l_0,l_0)}\right]\\ +1\cdot\left[\mathbf{\Psi}_{(l_0,l_0+1)}\cdot\mathbf{v}^{l_0}_{(l_0,l_0+1)}+\mathbf{\Psi}_{(l_0,l_0)}\cdot\mathbf{v}^{l_0}_{(l_0,l_0)}\right]\\ -1\cdot\left[\mathbf{\Psi}_{(l_0,l_0)}\cdot\mathbf{v}^{l_0}_{(l_0,l_0)}\right]\end{array}$$

which is the left hand side of (22).
For the proof of the general case, we compare the coefficients of $\mathbf{\Psi_m}$, $\mathbf{m}\in\Lambda$, in (22). Then, we have to show

$$\mathbf{v}_{\mathbf{m}}^{L(\hat{\mathbf{m}})}\stackrel{!}{=}\sum_{\mathbf{l}\in\Lambda}f(\mathbf{l})\sum_{\mathbf{k}\leq\mathbf{l}}\mathbf{v}_{\mathbf{k}}^{l_1}\delta(\mathbf{k}-\mathbf{m})\ . \tag{23}$$

There are two possibilities for the sum $\sum_{\mathbf{k}\leq\mathbf{l}}\mathbf{v}_{\mathbf{k}}^{l_1}\delta(\mathbf{k}-\mathbf{m})$:

$$\sum_{\mathbf{k}\leq\mathbf{l}}\mathbf{v}_{\mathbf{k}}^{l_1}\delta(\mathbf{k}-\mathbf{m})\ =\ \begin{cases}0 & \text{if } l_i<m_i \ \text{ for some }\ i\in\{1,...,d\}\ ,\\ \mathbf{v}_{\mathbf{m}}^{l_1} & \text{if } \mathbf{l}\geq\mathbf{m}\ .\end{cases}$$

Thus, the right-hand side of (23) reads:

$$\begin{aligned}\sum_{\mathbf{l}\geq\mathbf{m}}f(\mathbf{l})\mathbf{v}_{\mathbf{m}}^{l_1}\ &=\ \sum_{l_1=m_1}^{L(\hat{\mathbf{m}})}\mathbf{v}_{\mathbf{m}}^{l_1}\sum_{\hat{\mathbf{l}}\geq\hat{\mathbf{m}}}f((l_1,\hat{\mathbf{l}}))\\ &=\ \mathbf{v}_{\mathbf{m}}^{L(\hat{m})}\ .\end{aligned} \tag{24}$$

For the last step the following argument was used: Let $\Lambda^*:=\Lambda\bigcap\{\mathbf{l}\mid\mathbf{l}\geq\mathbf{m}\}$. Then, for the weights $f^*(.)$ according to Λ^* there holds

$$f^*(\mathbf{l})\ =\ f(\mathbf{l})\qquad\text{for}\quad \mathbf{l}\geq\mathbf{m}\ .$$

Furthermore, $L(\hat{\mathbf{m}})=lmax^*(1)$, and $\{\mathbf{l}\mid\hat{\mathbf{l}}\geq\hat{\mathbf{m}}\}=\Lambda^*(1,m_1)$. Thus, we can apply Proposition 3.1 and we obtain

$$\sum_{\mathbf{l}:\ \hat{\mathbf{l}}\geq\hat{\mathbf{m}}}f(\mathbf{l})\ =\ \begin{cases}1 & \text{if } l_1=L(\hat{\mathbf{m}})\ ,\\ 0 & \text{else}\ .\end{cases}$$

This proves the last step in (24) and, thus, the equivalence of D^Λ and **(FD)**.

5 Conclusion

In this paper, we have shown the consistency of the FD technique on regular sparse grids under mild conditions and for a general class of underlying FD schemes. To this end, we introduced an auxiliary discretization of differential operators which uses the ideas of the combination technique. It was shown that the auxiliary discretization coincides with the one of [7, 12, 18]. Thus, the above consistency results generalize and improve the results given in [18].

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References

[1] Bungartz H., *Dünne Gitter und deren Anwendung bei der adaptiven Lösung der 3D Poisson-Gleichung*; PhD thesis, TU München (1992)

[2] Bungartz H., *Finite elements of higher order on sparse grids*; Habil. thesis, TU München (1998)

[3] Liem C.B., Lue T., Shih T.M., *The splitting extrapolation method*; World Scientific, (1995) Singapore

[4] Daubechies I., *Ten lectures on wavelets*; CBMS-NSF Regional Conference Series in Appl. Math. 61 (1992)

[5] Deslaurier G., Dubuc S., *Symmetric iterative interpolation processes*; Constr. Approx. 5 (1989), pp. 49-68

[6] DeVore R., Konyagin S.V., Temlyakov V.N., *Hyperbolic wavelet approximation*; Constr. Approx. 14 (1998), pp. 1-26

[7] Griebel M., *Adaptive sparse grid multilevel methods for elliptic PDEs based on finite differences*; Computing 61 No. 2 (1998), pp. 151-180

[8] Griebel M., *A parallelizable and vectorizable multi-level algorithm on sparse grids*; in Hackbusch W. (ed.), Notes on Numerical Fluid Mechanics Vol. 31, Vieweg Verlag (1991)

[9] Griebel M., Knapek S., *Optimized approximation spaces for operator equations*; Preprint Universität Bonn (1998), accepted for publication in Constr. Approx.

[10] Griebel M., Oswald P., *Tensor product type subspace splitting and multilevel iterative methods for anisotropic problems*; Adv. Comp. Math. 4 (1995), pp. 171-206

[11] Hemker P., Pflaum C., *Approximation on partially ordered sets of regular grids*; Applied Numerical Mathematics: Trans. of IMACS, (1997)

[12] Hemker P., Sprengel F., *On the representation of functions and finite difference operators on adaptive sparse grids*; Report CWI (Netherlands) (1999)

[13] Griebel M., Koster F., *Adaptive Wavelet Solvers for the Unsteady Incompressible Navier-Stokes Equations*; to appear in Malek J., Rokyta M. (eds.), Advanced Mathematical Theories in Fluid Mechanics, Springer (2000)

[14] Noordmans J., Hemker P., *Application of an adaptive sparse grid technique to a model singular perturbation problem*; Report CWI (Netherlands) (1999)

[15] Pflaum, C., *Diskretisierung elliptischer Differentialgleichungen mit dünnen Gittern*; PhD thesis, TU München (1996)

[16] Pflaum, C., *Convergence of the combination technique for the solution of second-order elliptic differential equations*; SIAM J. Numer. Anal. 34 (1997), pp. 2431-2455

[17] Rüde U., *Multilevel extrapolation and sparse grid methods*; in Hemker P.W. (ed.), Multigrid methods IV ISNM 116, Birkhäuser Verlag (1994)

[18] Schiekofer T., *Die Methode der finiten Differenzen auf dünnen Gittern zur Lösung elliptischer und parabolischer PDEs*; PhD thesis, Universität Bonn (1998)

[19] Shu C.-W., *Higher Order ENO and WENO Schemes for Computational Fluid Dynamics*; in Barth T., Deconinck H. (eds.), Higher Order Methods for Computational Physics, LNCSE series Vol 9, Springer Verlag (1999)

[20] Sprengel F., *A Tool for Approximation in Bivariate Periodic Sobolev Spaces*; in Chui C., Schumaker L. (eds.), Approximation Theory IX Vol. 2, Vanderbuilt Univ. Press, Nashville (1999), pp. 319-326

[21] Temlyakov V., *Approximation of periodic functions*; Nova Science Publ. New York (1994)

[22] Triebel H., *Theory of function spaces II*; Birkhäuser Verlag (1992)

[23] Zenger C., *Sparse Grids*; in Hackbusch W. (ed.), Notes on Numerical Fluid Mechanics Vol. 31, Vieweg Verlag (1991)

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