POINTWISE CONVERGENCE OF THE COMBINATION TECHNIQUE FOR LAPLACE'S EQUATION [1]

H.-J. BUNGARTZ, M. GRIEBEL, D. RÖSCHKE, AND C. ZENGER [2]

Abstract. For a simple model problem — the Laplace equation on the unit square with a Dirichlet boundary function vanishing for $x = 0$, $x = 1$, and $y = 1$, and equaling some suitable $g(x)$ for $y = 0$ — we present a proof of convergence for the combination technique, a modern, efficient, and easily parallelizable sparse grid solver for elliptic partial differential equations that recently gained importance in fields of applications like computational fluid dynamics. For full square grids with meshwidth h and $O(h^{-2})$ grid points, the order $O(h^2)$ of the discretization error using finite differences was shown in **[12]**, if $g(x) \in C^2[0,1]$. In this paper, we show that the finite difference discretization error of the solution produced by the combination technique on a sparse grid with only $O\left((h^{-1}\log_2(h^{-1})\right)$ grid points is of the order $O\left(h^2\log_2(h^{-1})\right)$, if the Fourier coefficients b_k of $\tilde{g}$, the 2-periodic and 0-symmetric extension of g, fulfill $|b_k| \le c_g \cdot k^{-3-\varepsilon}$ for some arbitrary small positive ε. If $0 < \varepsilon \le 1$, this is vaild for $g \in C^4[0,1]$ and $g(0) = g(1) = g''(0) = g''(1) = 0$, e.g.. A simple transformation even shows that $g \in C^4[0,1]$ is sufficient. Furthermore, we present results of numerical experiments with functions g of varying smoothness.

Key words. Combination technique, sparse grids, elliptic partial differential equations, order of discretization error.

AMS(MOS) subject classifications. 35J05, 65N06, 65N15, 65N22, 65N55.

1. Introduction.

Since their presentation in 1990 **[17]**, sparse grids have turned out to be a very interesting tool for the efficient solution of elliptic boundary value problems. Besides the implementation of a hierarchical finite element algorithm on sparse grids **[1]**, it has been primarily the combination technique **[9]** that attained attention in the sparse grid context. Here, first, the problem is solved on several rectangular grids with different meshwidths h_x and h_y in x- and y-direction, respectively. Afterwards, the computed solutions are combined linearly in order to get the desired sparse grid solution. Consequently, one of the main advantages of the combination technique stems from the properties of sparse grids **[1]**: In comparison to the standard full grid approach, the number of grid points can be reduced significantly from $O(N^d)$ to $O(N(\log_2(N))^{d-1})$ in the d-dimensional case, whereas the accuracy of the calculated approximation to the solution is only slightly deteriorated from $O(N^{-2})$ (see **[12]**) to $O(N^{-2}(\log_2(N))^{d-1})$. Another advantage has to be seen in the simplicity of the combination concept, its inherent parallel structure, and its framework property allowing the integration of existing solvers for partial differential equations.

Up to now, the combination technique has been applied to several types of elliptic partial differential equations including problems originating from computational fluid dynamics. Its excellent parallelization properties have been verified on different parallel architectures and on workstation networks **[5, 6, 7, 8, 10]**.

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[2] Institut für Informatik der Technischen Universität München, D-80290 München, Germany.

For a proof of the approximation properties mentioned above, in the two-dimensional case the existence of an error splitting

$$(1) \quad u_{h_x,h_y}(x,y) - u(x,y) \;=\; C_1(x,y,h_x)h_x^2 + C_2(x,y,h_y)h_y^2 + C(x,y,h_x,h_y)h_x^2h_y^2$$

with $|C_1(x,y,h_x)|$, $|C_2(x,y,h_y)|$, and $|C(x,y,h_x,h_y)|$ bounded by some positive $B(x,y)$ for all meshwidths h_x and h_y has to be shown **[9]**. Here, $u(x,y)$ is the exact solution of the given boundary value problem, and $u_{h_x,h_y}(x,y)$ denotes the solution on the rectangular full grid with meshwidths h_x and h_y resulting from a finite difference discretization. For a simple model problem — the Laplace equation on the unit square with a Dirichlet boundary function vanishing for $x = 0$, $x = 1$, and $y = 1$, and equaling some suitable $g(x)$ for $y = 0$ — we present a proof for the existence of such an error splitting, if g satisfies a certain smoothness requirement: The Fourier coefficients b_k of the 2-periodic and 0-symmetric extension of g have to be of the order $O(k^{-3-\varepsilon})$ for some arbitrary small positive ε.

In section 2, we give a short survey of the combination technique and its most important properties. Then, the following sections deal with the proof of (1). Since, for the error splitting presented in section 6, we need explicit representations of u, u_{h_x,h_y}, and of the solutions $u_{h_x,0}$ and u_{0,h_y} of the Laplacian discretized in only one direction, a Fourier-based representation of those functions is given in section 3. After showing some properties of the components of u, u_{h_x,h_y}, $u_{h_x,0}$, and u_{0,h_y} in sections 4 and 5, we are able to define an error splitting according to (1). Finally, in section 7, we present numerical examples with Dirichlet boundary functions of varying smoothness.

2. The Combination Technique. Let L be an elliptic operator of second order. Consider the partial differential equation $Lu = f$ on the unit square $\Omega :=]0,1[^2$ with appropriate boundary conditions. Furthermore, let $G_{i,j}$ be the rectangular grid on $\bar{\Omega}$ with meshwidths $h_x := h_i := 2^{-i}$ in x- and $h_y := h_j := 2^{-j}$ in y-direction ($i, j \in \mathbb{N}$). In **[9]**, the combination technique was introduced, which combines the solutions u_{h_x,h_y} of the discrete problems $L_{h_x,h_y}u_{h_x,h_y} = f_{h_x,h_y}$ associated to $Lu = f$ on different rectangular grids $G_{i,j}$:

$$(2) \qquad u_n^c := \sum_{i+j=n+1} u_{h_i,h_j} - \sum_{i+j=n} u_{h_i,h_j}.$$

The resulting solution u_n^c is given on the sparse grid $\tilde{G}_{n,n}$ with $O(2^n n)$ grid points instead of the usual full grid $G_{n,n}$ with $O(4^n)$ points. Figure 1 shows the sparse grids $\tilde{G}_{3,3}$ and $\tilde{G}_{4,4}$. For a detailed introduction to sparse grids, see **[17]** and **[1]**.

Note that we have to solve n different problems with $O(2^{n+1})$ grid points ($i + j = n + 1$) and $n - 1$ different problems with $O(2^n)$ grid points ($i + j = n$). These $2n - 1$ problems are independent from one another and can be solved in parallel. Finally, their bilinearly interpolated solutions are combined according to (2).

Concerning the error $e_n^c(x,y)$ of the combined solution $u_n^c(x,y)$, we cite a result from **[9]**. To complete the picture, we give the proof, too.

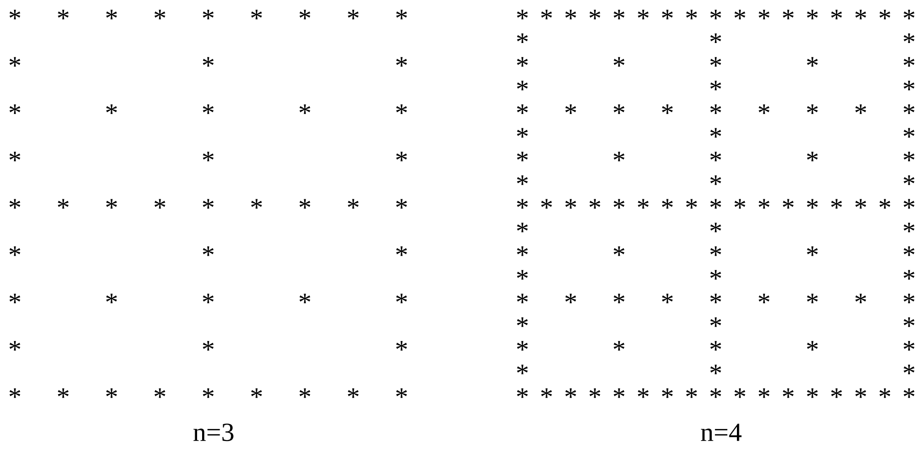

FIG. 1. *Sparse grids* $\tilde{G}_{3,3}$ *and* $\tilde{G}_{4,4}$.

THEOREM 1. *Assume that, for each inner point* $(x, y) \in \Omega$, *the pointwise error of the solution* $u_{h_x,h_y} = u_{2^{-i},2^{-j}}$ *obtained on the rectangular grid* $G_{i,j}$ *with meshwidths* $h_x = h_i = 2^{-i}$ *and* $h_y = h_j = 2^{-j}$ *is of the form (1) with* $|C_1(x, y, h_x)|$, $|C_2(x, y, h_y)|$, *and* $|C(x, y, h_x, h_y)|$ *bounded by some positive* $B(x, y)$ *for all* $h_x, h_y \in \,]0, \frac{1}{2}]$. *Then, the error* $e_n^c(x, y)$ *of the combined solution* $u_n^c(x, y)$ *on the sparse grid* $\tilde{G}_{n,n}$ *fulfills*

$$|e_n^c(x, y)| \;:=\; |u_n^c(x, y) - u(x, y)| \;\leq\; B(x, y) \cdot h_n^2 \cdot \Big(1 + \frac{5}{4} \cdot \log_2(h_n^{-1})\Big), \tag{3}$$

where $h_n = 2^{-n}$.

Proof. Looking at definition (2), we have

$$\begin{aligned} e_n^c(x, y) &= u_n^c(x, y) - u(x, y) \\ &= \sum_{i+j=n+1} u_{h_i,h_j}(x, y) \;-\; \sum_{i+j=n} u_{h_i,h_j}(x, y) \;-\; u(x, y) \\ &= \sum_{i+j=n+1} \Big(u_{h_i,h_j}(x, y) - u(x, y)\Big) \;-\; \sum_{i+j=n} \Big(u_{h_i,h_j}(x, y) - u(x, y)\Big), \end{aligned}$$

and, assuming (1), we get

$$\begin{aligned} e_n^c(x, y) \;=\; & \sum_{i+j=n+1} \Big(C_1(x, y, h_i) \cdot h_i^2 \;+\; C_2(x, y, h_j) \cdot h_j^2 \;+\; C(x, y, h_i, h_j) \cdot h_i^2 h_j^2\Big) \\ & - \sum_{i+j=n} \Big(C_1(x, y, h_i) \cdot h_i^2 \;+\; C_2(x, y, h_j) \cdot h_j^2 \;+\; C(x, y, h_i, h_j) \cdot h_i^2 h_j^2\Big) \\ =\; & \sum_{i=1}^{n} C_1(x, y, h_i) \cdot h_i^2 \;-\; \sum_{i=1}^{n-1} C_1(x, y, h_i) \cdot h_i^2 \\ & + \sum_{j=1}^{n} C_2(x, y, h_j) \cdot h_j^2 \;-\; \sum_{j=1}^{n-1} C_2(x, y, h_j) \cdot h_j^2 \\ & + \sum_{i+j=n+1} C(x, y, h_i, h_j) \cdot h_i^2 h_j^2 \;-\; \sum_{i+j=n} C(x, y, h_i, h_j) \cdot h_i^2 h_j^2 \end{aligned}$$

$$
\begin{aligned}
&= C_1(x,y,h_n)\cdot h_n^2 \;+\; C_2(x,y,h_n)\cdot h_n^2 \\
&\quad + \left(\frac{1}{4}\cdot \sum_{i+j=n+1} C(x,y,h_i,h_j) \;-\; \sum_{i+j=n} C(x,y,h_i,h_j)\right)\cdot h_n^2.
\end{aligned}
$$

This leads to the estimation

$$
\begin{aligned}
|e_n^c(x,y)| &\leq |C_1(x,y,h_n)|\cdot h_n^2 \;+\; |C_2(x,y,h_n)|\cdot h_n^2 \\
&\quad + \left(\frac{1}{4}\cdot \sum_{i+j=n+1} |C(x,y,h_i,h_j)| \;+\; \sum_{i+j=n} |C(x,y,h_i,h_j)|\right)\cdot h_n^2 \\
&\leq B(x,y)\cdot h_n^2\cdot \left(2+\frac{n}{4}+n-1\right) \\
&= B(x,y)\cdot h_n^2\cdot \left(1+\frac{5}{4}\cdot \log_2(h_n^{-1})\right).
\end{aligned}
$$

□

Thus, if (1) holds, the pointwise error of the combined solution is of the order $O(h_n^2 \log_2(h_n^{-1}))$, but the computation of u_n^c only involves the $O(h_n^{-1}\log_2(h_n^{-1}))$ grid points of the sparse grid $\tilde{G}_{n,n}$. In other words, on a sparse grid with significantly less grid points than on a standard full grid, the combination technique produces a solution that is nearly as good as a full grid solution.

In this paper, we deal with the question which smoothness requirements have to be fulfilled by the Dirichlet boundary function in order to get a behaviour of the error $u_{h_x,h_y}(x,y)-u(x,y)$ on $G_{i,j}$ as indicated in (1).

For the remainder, we concentrate on the Laplace equation

$$
\Delta u(x,y) = 0, \qquad (x,y)\in\Omega, \tag{4}
$$

with the Dirichlet boundary condition

$$
u(x,y) = \begin{cases} g(x) : (x,y)\in\delta\bar{\Omega}, y=0 \\ \;0 \quad\; : (x,y)\in\delta\bar{\Omega}, y>0. \end{cases} \tag{5}
$$

Throughout the paper, we assume that the Fourier coefficients b_k of $\tilde{g}$, the 2-periodic and 0-symmetric extension of g, fulfill

$$
|b_k| \;:=\; \left|\int_{-1}^{1}\tilde{g}(x)\sin(k\pi x)dx\right| \;=\; \left|2\int_0^1 g(x)\sin(k\pi x)dx\right| \;\leq\; \frac{c_g}{k^{3+\varepsilon}} \tag{6}
$$

for some arbitrary small positive ε and a constant c_g depending only on g. It is a well-known fact from Fourier theory that this is valid with $0<\varepsilon\leq 1$ if $\tilde{g}\in C^4(\mathbb{R})$ or if $g\in C^4[0,1]$ and $g(0)=g(1)=g''(0)=g''(1)=0$.

3. Explicit Solutions. As a first step on the way to a proof of (1), we introduce a Fourier-based representation of u, u_{h_x,h_y}, $u_{h_x,0}$, and u_{0,h_y}. Here, as mentioned before, $u_{h_x,0}$ and u_{0,h_y} denote the solutions of the Laplacian discretized only in x- or y-direction, respectively. These Fourier series will enable us to find an appropriate splitting of $u_{h_x,h_y}-u$ in section 6.

3.1. Continuous Problem. For an explicit representation of the solution $u(x,y)$ of (4) and (5), we cite a result from **[16]**:

THEOREM 2.

$$u(x,y) \;:=\; \sum_{k=1}^{\infty} b_k \sin(k\pi x) T(k,y) \tag{7}$$

with b_k defined in (6) and

$$T(k,y) \;:=\; \frac{\sinh(k\pi(1-y))}{\sinh(k\pi)} \tag{8}$$

solves (4) and satisfies the boundary condition (5).

3.2. Discrete Problem. In contrast to the usual square grid approach, we want to allow nets for the discretization with different meshwidths h_x and h_y in x- and in y-direction. Therefore, the well-known five-point-stencil normally used in connection with finite differences has to be modified. The discrete counterpart to the continuous problem then reads

$$\begin{aligned} \Delta_{h_x,h_y} u_{h_x,h_y}(x,y) &= \frac{u_{h_x,h_y}(x+h_x,y) + u_{h_x,h_y}(x-h_x,y) - 2u_{h_x,h_y}(x,y)}{h_x^2} \\ &+ \frac{u_{h_x,h_y}(x,y+h_y) + u_{h_x,h_y}(x,y-h_y) - 2u_{h_x,h_y}(x,y)}{h_y^2} \\ &= 0, \end{aligned} \tag{9}$$

where

$$\begin{aligned} x = ih_x,\;\; 1 \le i \le N_x - 1,\;\; h_x = 1/N_x,\;\; N_x \in \mathbb{N} \setminus \{1\}\,, \\ y = jh_y,\;\; 1 \le j \le N_y - 1,\;\; h_y = 1/N_y,\;\; N_y \in \mathbb{N} \setminus \{1\}\,. \end{aligned}$$

In the grid points on the boundary $\delta\bar{\Omega}$, we postulate

$$u_{h_x,h_y}(ih_x, jh_y) \;=\; \begin{cases} g(ih_x) & : \;\; (ih_x, jh_y) \in \delta\bar{\Omega},\; j = 0 \\ 0 & : \;\; (ih_x, jh_y) \in \delta\bar{\Omega},\; j > 0. \end{cases} \tag{10}$$

On the analogy of the continuous case, we call functions satisfying (9) *discrete harmonic*. Again, we present a Fourier-type representation of a function $u_{h_x,h_y}(x,y)$ that solves (9) and fulfills (10):

THEOREM 3.

$$u_{h_x,h_y}(x,y) \;:=\; \sum_{k=1}^{\infty} b_k \sin(k\pi x) T(\mu_k, y) \tag{11}$$

with

$$\frac{1-\cos(k\pi h_x)}{h_x^2} \;=\; \frac{\cosh(\mu_k \pi h_y) - 1}{h_y^2}, \tag{12}$$

defining μ_k implicitly as a function of k, h_x, and h_y, is discrete harmonic and even satisfies the continuous boundary condition (5). For the definitions of b_k and $T(\mu_k, y)$, see (6) and (8), respectively.

Proof. The proof follows **[12]**. In (9), we use the separation approach

$$u_{h_x,h_y}(x,y) \;=\; X(x)\cdot Y(y),$$

and get

$$Y(y)\frac{X(x+h_x)+X(x-h_x)-2X(x)}{h_x^2} \;+\; X(x)\frac{Y(y+h_y)+Y(y-h_y)-2Y(y)}{h_y^2} \;=\; 0.$$

Note that the following relations hold:

$$\sin\Big(k\pi(x+h_x)\Big) \;+\; \sin\Big(k\pi(x-h_x)\Big) \;=\; 2\sin(k\pi x)\cdot\cos(k\pi h_x)$$

and

$$\sinh\Big(\mu\pi(1-y+h_y)\Big) \;+\; \sinh\Big(\mu\pi(1-y-h_y)\Big) \;=\; 2\sinh\Big(\mu\pi(1-y)\Big)\cdot\cosh(\mu\pi h_y).$$

These considerations suggest the choice

$$u_{h_x,h_y}(x,y) \;=\; \sin(k\pi x)\cdot\sinh\left(\mu\pi(1-y)\right), \tag{13}$$

and it can be shown, indeed, that for each $k \in \mathbb{N}$ $u_{h_x,h_y}(x,y)$ given in (13) is solution of (9) on condition that

$$\frac{1-\cos(k\pi h_x)}{h_x^2} \;=\; \frac{\cosh(\mu\pi h_y)-1}{h_y^2}.$$

So, relation (12) has been shown. To indicate the dependence on k, we are going to write μ_k instead of μ from now on.

Since the discrete operator Δ_{h_x,h_y} is linear,

$$\sum_{k=1}^{\infty}\beta_k\sin(k\pi x)\sinh\Big(\mu_k\pi(1-y)\Big) \tag{14}$$

with arbitrary $\beta_k \in \mathbb{R}$ solves (9), too, if we assure convergence. Note that (14) already satisfies the continuous boundary condition given in (5) for $y > 0$. For $y = 0$, we request that our solution also satisfies the according boundary condition from (5):

$$\sum_{k=1}^{\infty}\beta_k\sinh(\mu_k\pi)\sin(k\pi x) \;=\; g(x), \quad 0 \le x \le 1. \tag{15}$$

Fourier theory and comparison of the coefficients provide

$$\beta_k\sinh(\mu_k\pi) = b_k.$$

The resulting series given in (11) converges due to (6) and because $|T(t,y)| \le 1$. This completes our proof. □

3.3. Discrete Problem in x-, Continuous Problem in y-Direction. Discretizing the Laplacian only in x-direction leads to

$$\begin{aligned}\Delta_{h_x,0}u_{h_x,0}(x,y) &= \frac{u_{h_x,0}(x+h_x,y)+u_{h_x,0}(x-h_x,y)-2u_{h_x,0}(x,y)}{h_x^2} \\ &+ \frac{\partial^2 u_{h_x,0}(x,y)}{\partial y^2} \\ &= 0,\end{aligned} \tag{16}$$

where

$$\begin{aligned}&x = ih_x, \quad 1 \le i \le N_x - 1, \quad h_x = 1/N_x, \quad N_x \in \mathbb{N}\setminus\{1\}, \\ &y \in \;]0,1[.\end{aligned}$$

In the grid points on the boundary $\delta\bar{\Omega}$, we postulate

$$u_{h_x,0}(ih_x,y) = \begin{cases} g(ih_x) & : \quad (ih_x,y) \in \delta\bar{\Omega}, \; y=0 \\ 0 & : \quad (ih_x,y) \in \delta\bar{\Omega}, \; y>0. \end{cases} \tag{17}$$

The limits for $h_y \to 0$ from equations (11) and (12) suggest

THEOREM 4.

$$u_{h_x,0}(x,y) := \sum_{k=1}^{\infty} b_k \sin(k\pi x) T(\nu_k, y) \tag{18}$$

with

$$\frac{\nu_k^2\pi^2}{2} = \frac{1-\cos(k\pi h_x)}{h_x^2}, \tag{19}$$

defining ν_k as a function of k and h_x, solves equation (16) and even satisfies the continuous boundary condition (5).

Proof. The convergence of the series in (18) is, again, assured by (6). Using the separation approach

$$u_{h_x,0}(x,y) = X(x)\cdot Y(y)$$

in equation (16), we get

$$\Delta_{h_x,0}u_{h_x,0}(x,y) = \frac{X(x+h_x)+X(x-h_x)-2X(x)}{h_x^2}Y(y) + X(x)Y''(y) = 0. \tag{20}$$

With our proposed solutions $X(x) = \sin(k\pi x)$ and $Y(y) = T(\nu_k, y)$, we obtain a direct verification of equation (20):

$$\Delta_{h_x,0}u_{h_x,0}(x,y) = \frac{2\sin(k\pi x)\cos(k\pi h_x) - 2\sin(k\pi x)}{h_x^2}\cdot\frac{\sinh(\nu_k\pi(1-y))}{\sinh(\nu_k\pi)}$$

$$
\begin{aligned}
&+ \quad \sin(k\pi x) \cdot \frac{\nu_k^2 \pi^2 \sinh(\nu_k \pi (1-y))}{\sinh(\nu_k \pi)} \\
&= \quad 2\sin(k\pi x) \cdot T(\nu_k, y) \cdot \left(\frac{\cos(k\pi h_x) - 1}{h_x^2} + \frac{\nu_k^2 \pi^2}{2} \right) \\
&= \quad 0,
\end{aligned}
$$

if condition (19) holds.

Our solution (18) satisfies the boundary condition (5) for $y > 0$. For $y = 0$, we get

$$u_{h_x,0}(x, 0) \;=\; \sum_{k=1}^{\infty} b_k \sin(k\pi x),$$

which is exactly the Fourier series of $g(x)$. Thus, again even the continuous boundary condition (5) are fulfilled, which completes our proof. □

3.4. Continuous Problem in x-, Discrete Problem in y-Direction. This time, we discretize the Laplacian only in y-direction:

$$
\begin{aligned}
\Delta_{0,h_y} u_{0,h_y}(x,y) \;&=\; \frac{\partial^2 u_{0,h_y}(x,y)}{\partial x^2} \\
&+\; \frac{u_{0,h_y}(x, y+h_y) + u_{0,h_y}(x, y-h_y) - 2u_{0,h_y}(x,y)}{h_y^2} \\
&=\; 0,
\end{aligned}
\tag{21}
$$

where

$$
\begin{aligned}
& x \in \,]0,1[, \\
& y = jh_y, \quad 1 \le j \le N_y - 1, \;\; h_y = 1/N_y, \;\; N_y \in \mathbb{N} \setminus \{1\}\,.
\end{aligned}
$$

In the grid points on the boundary $\delta\bar{\Omega}$, we postulate

$$
u_{0,h_y}(x, jh_y) \;=\; \begin{cases} g(x) & : \;\; (x, jh_y) \;\in\; \delta\bar{\Omega}, \;\; j = 0 \\ 0 & : \;\; (x, jh_y) \;\in\; \delta\bar{\Omega}, \;\; j > 0. \end{cases}
\tag{22}
$$

The limits for $h_x \to 0$ from equations (11) and (12) suggest

THEOREM 5.

$$u_{0,h_y}(x,y) \;:=\; \sum_{k=1}^{\infty} b_k \sin(k\pi x) T(\lambda_k, y) \tag{23}$$

with

$$\frac{k^2\pi^2}{2} \;=\; \frac{\cosh(\lambda_k \pi h_y) - 1}{h_y^2}, \tag{24}$$

defining λ_k implicitly as a function of k and h_y, solves equation (21) and even satisfies the continuous boundary condition (5).

We omit the proof, because it exactly follows the argumentation in the proof of theorem 4.

4. Properties of ν_k, λ_k, and μ_k. When we compare the explicit representations (7), (11), (18), and (23) of u, u_{h_x,h_y}, $u_{h_x,0}$, and u_{0,h_y}, respectively, we see that there is only one difference, namely the first argument of the function T. Therefore, to prepare the discussion of u, u_{h_x,h_y}, $u_{h_x,0}$, and u_{0,h_y}, it is important to study both those different first arguments $\mu_k(h_x, h_y)$, $\nu_k(h_x)$, and $\lambda_k(h_y)$ and the function T itself.

4.1. Properties of ν_k. Since $T(t, y)$ is an even function of t for every y, we can choose the following definition for ν_k instead of condition (19):

$$\frac{\nu_k \pi h_x}{2} \;=\; \sin\left(\frac{k\pi h_x}{2}\right). \tag{25}$$

This leads to

LEMMA 1. *For all $k \in \mathbb{N}$ with $k < N_x := \frac{1}{h_x}$, ν_k satisfies the estimates*

$$\frac{2}{\pi}k < \nu_k < k. \tag{26}$$

Here, the upper bound k for ν_k is valid for all $k \in \mathbb{N}$. Furthermore, $\lim_{h_x\to 0} \nu_k = k$ for all $k \in \mathbb{N}$.

Proof. From (25), we get

$$\nu_k - k \;=\; k \cdot \frac{\sin\left(\frac{k\pi h_x}{2}\right) - \frac{k\pi h_x}{2}}{\frac{k\pi h_x}{2}} \;=\; k \cdot \frac{\sin(w) - w}{w},$$

where $w := \frac{k\pi h_x}{2}$. For arbitrary $k \in \mathbb{N}$, we have $w \in \,]0, \infty[$, $\frac{\sin(w)-w}{w} < 0$, and, thus, $\nu_k < k$. If $k < N_x = \frac{1}{h_x}$, we get $w \in \,]0, \frac{\pi}{2}[$, $\frac{\sin(w)-w}{w} \in \,]\frac{2-\pi}{\pi}, 0[$, and, therefore, $\frac{2}{\pi}k < \nu_k$. Finally, $\lim_{h_x\to 0} \nu_k = k$ follows directly from (25). ☐

Next, we make a statement concerning the difference $\nu_k - k$:

LEMMA 2. *For all $k \in \mathbb{N}$, we have*

$$\nu_k - k \;=\; k^3 h_x^2 c_1(k, h_x), \tag{27}$$

where $|c_1(k, h_x)| < \frac{\pi^2}{24}$ and $\lim_{h_x\to 0} c_1(k, h_x) = -\frac{\pi^2}{24}$.

Proof. According to (25) and (27), $c_1(k, h_x)$ can be written as

$$\begin{aligned} c_1(k, h_x) \;:=\; \frac{\nu_k - k}{k^3 h_x^2} \;&=\; \frac{1}{k^3 h_x^2} \cdot \left(\frac{\sin\left(\frac{k\pi h_x}{2}\right)}{\frac{\pi h_x}{2}} - k\right) \\ &=\; \frac{\sin\left(\frac{k\pi h_x}{2}\right) - \frac{k\pi h_x}{2}}{\frac{k^3 \pi h_x^3}{2}} \\ &=\; \frac{\sin(w) - w}{w^3} \cdot \frac{\pi^2}{4} \end{aligned}$$

with $w := \frac{k\pi h_x}{2}$, showing both $|c_1(k, h_x)| < \frac{\pi^2}{24}$ and $\lim_{h_x\to 0} c_1(k, h_x) = -\frac{\pi^2}{24}$. ☐

4.2. Properties of $\boldsymbol{\lambda_k}$. Since $T(t,y)$ is an even function in its first argument, we assume that $\lambda_k > 0$ without loss of generality. Then, relation (24) is equivalent to

$$\frac{k\pi h_y}{2} \;=\; \sqrt{\frac{\cosh(\lambda_k \pi h_y) - 1}{2}} \;=\; \sinh\left(\frac{\lambda_k \pi h_y}{2}\right). \tag{28}$$

This leads to

LEMMA 3. *For all $k \in \mathbb{N}$, λ_k satisfies the estimates*

$$\frac{4}{\pi}\ln\left(\frac{k\pi}{2}\right) \;<\; \lambda_k \;<\; k, \tag{29}$$

and $\lim_{h_y\to 0}\lambda_k = k$.

Proof. $\lambda_k < k$ immediately follows from (28), because $\sinh(x) > x$ for all $x > 0$. On the other hand, let us assume that $\lambda_k \le \frac{4}{\pi}\ln\left(\frac{k\pi}{2}\right)$:

$$\lambda_k \le \frac{4}{\pi}\ln\left(\frac{k\pi}{2}\right) \;\Rightarrow\; \sinh\left(\frac{\lambda_k\pi}{4}\right) \le \sinh\left(\ln\left(\frac{k\pi}{2}\right)\right) < \frac{1}{2}\exp\left(\ln\left(\frac{k\pi}{2}\right)\right) = \frac{k\pi}{4},$$

which contradicts (28) for $h_y = \frac{1}{2}$. Finally, $\lim_{h_y\to 0}\lambda_k = k$ follows directly from (28). $\square$

Now, we have a closer look on the difference $\lambda_k - k$:

LEMMA 4. *For all $k \in \mathbb{N}$, we have*

$$\lambda_k - k = k^3 h_y^2 c_2(k, h_y), \tag{30}$$

where $|c_2(k,h_y)| < \frac{\pi^2}{24}$ *and* $\lim_{h_y\to 0} c_2(k,h_y) = -\frac{\pi^2}{24}$.

Proof. According to (28) and (30), $c_2(k,h_y)$ can be written as

$$\begin{aligned} c_2(k,h_y) \;:=\; \frac{\lambda_k - k}{k^3 h_y^2} \;&=\; \frac{1}{k^3 h_y^2}\cdot\left(\frac{\operatorname{arsinh}\left(\frac{k\pi h_y}{2}\right)}{\frac{\pi h_y}{2}} - k\right) \\ &=\; \frac{\operatorname{arsinh}\left(\frac{k\pi h_y}{2}\right) - \frac{k\pi h_y}{2}}{\frac{k^3\pi h_y^3}{2}} \\ &=\; \frac{\operatorname{arsinh}(w) - w}{w^3}\cdot\frac{\pi^2}{4} \end{aligned}$$

with $w := \frac{k\pi h_y}{2}$, showing both $|c_2(k,h_y)| < \frac{\pi^2}{24}$ and $\lim_{h_y\to 0} c_2(k,h_y) = -\frac{\pi^2}{24}$. $\square$

4.3. Properties of $\boldsymbol{\mu_k}$. Again, we profit from the fact that $T(t,y)$ is an even function of t. Then, relations (12) and (19) lead to

$$\frac{\nu_k \pi h_y}{2} \;=\; \sinh\left(\frac{\mu_k \pi h_y}{2}\right). \tag{31}$$

Thus, we get

LEMMA 5. *For all $k \in \mathbb{N}$ with $k < N_x$, μ_k satisfies the estimates*

$$\frac{4}{\pi}\ln\left(\frac{k\pi}{4}\right) \;<\; \mu_k \;<\; k. \tag{32}$$

Here, the upper bound k for μ_k is valid for all $k \in \mathbb{N}$. Furthermore, $\lim_{h_y\to 0}\mu_k = \nu_k$ and $\lim_{h_x\to 0}\mu_k = \lambda_k$ for all $k \in \mathbb{N}$.

Proof. Applying the argumentation of lemma 3 to (31), we immediately get

$$\frac{4}{\pi}\ln\left(\frac{\nu_k\pi}{2}\right) \;<\; \mu_k \;<\; \nu_k.$$

Together with the bounds for ν_k from lemma 1, we obtain (32). Finally, $\lim_{h_y\to 0}\mu_k = \nu_k$ follows directly from (31), and $\lim_{h_x\to 0}\mu_k = \lambda_k$ results from (28), (31), and the fact that $\lim_{h_x\to 0}\nu_k = k$. □

5. Properties of $T(t,y)$. In this section, we finish the discussion of u, u_{h_x,h_y}, $u_{h_x,0}$, and u_{0,h_y} with some remarks on important properties of $T(t,y)$.

LEMMA 6. *Let $y \in]0,1[$ be arbitrary, but fixed. With $T(0,y) := 1-y$, $T(t,y)$ becomes an even function of t continuous and differentiable for all $t \in \mathbb{R}$, and*

$$T(t,y) \;\le\; \min\left\{\mathrm{e}^{-|t|\pi y}, 1-y\right\}. \tag{33}$$

Furthermore,

$$\frac{\partial T(t,y)}{\partial t} \;=\; T(t,y)\cdot P(t,y) \;=:\; T(t,y)\cdot\pi\cdot\Big((1-y)\coth(t\pi(1-y)) - \coth(t\pi)\Big), \tag{34}$$

where

$$|P(t,y)| \;\le\; \pi y. \tag{35}$$

Proof. We learn from definition (8) that $T(t,y)$ is an even function in its first argument. Since

$$\lim_{t\to 0} T(t,y) \;=\; \lim_{t\to 0}\frac{\sinh(t\pi(1-y))}{\sinh(t\pi)} \;=\; (1-y)\lim_{t\to 0}\frac{\cosh(t\pi(1-y))}{\cosh(t\pi)} \;=\; 1-y,$$

$T(t,y)$ can be extended continuously in $t=0$. Furthermore, $T(t,y)$ is differentiable in $t=0$:

$$\left.\frac{\partial T(t,y)}{\partial t}\right|_{t=0} \;=\; \lim_{t\to 0}\frac{T(t,y)-T(0,y)}{t} \;=\; \lim_{t\to 0}\frac{\sinh(t\pi(1-y)) - (1-y)\sinh(t\pi)}{t\sinh(t\pi)} \;=\; 0,$$

if we apply L'Hospital's rule twice.

Next, we turn to the bounds for $T(t,y)$. As $T(t,y)$ is positive and even in t, we only have to show that $T(t,y)$ decreases monotonously for $t > 0$. Consider

$$\frac{\partial T(t,y)}{\partial t} \;=\; \frac{\pi(1-y)\sinh(t\pi)\cosh(t\pi(1-y)) \;-\; \pi\sinh(t\pi(1-y))\cosh(t\pi)}{(\sinh(t\pi))^2}.$$

Consequently,

$$\begin{aligned}\frac{\partial T(t,y)}{\partial t} < 0 \quad &\Leftrightarrow \quad (1-y)\cosh(t\pi(1-y))\sinh(t\pi) < \sinh(t\pi(1-y))\cosh(t\pi) \\ &\Leftrightarrow \quad (1-y)\tanh(t\pi) < \tanh(t\pi(1-y)) \\ &\Leftrightarrow \quad y\tanh(0) + (1-y)\tanh(t\pi) < \tanh(y0 + (1-y)t\pi),\end{aligned}$$

which is fulfilled due to the fact that tanh is strictly concave in $(0,\infty)$. Thus, $T(t,y) \le T(0,y) = 1-y$ for all $t \in \mathbb{R}$. Furthermore, we have

$$T(t,y) = T(|t|,y) = \frac{\sinh(|t|\pi(1-y)}{\sinh(|t|\pi)} = \mathrm{e}^{-|t|\pi y}\frac{1-\mathrm{e}^{-2|t|\pi(1-y)}}{1-\mathrm{e}^{-2|t|\pi}} < \mathrm{e}^{-|t|\pi y},$$

which completes the proof of (33).

Since (34) can be easily verified, we finally show (35). First of all, note that $P(t,y)$ is an odd function in t with $\lim_{t\to\infty} P(t,y) = -\pi y$ due to $\lim_{t\to\infty}\coth(t) = 1$ and $\lim_{t\to 0} P(t,y) = 0$ because $\frac{\partial T(t,y)}{\partial t}$ vanishes for $t = 0$. Furthermore, $P(t,y)$ decreases monotonously for $t > 0$:

$$\begin{aligned}\frac{\partial P(t,y)}{\partial t} < 0 \quad &\Leftrightarrow \quad \pi\Big(-\pi(1-y)^2\frac{1}{\sinh^2(t\pi(1-y))} + \pi\frac{1}{\sinh^2(t\pi)}\Big) \ < \ 0 \\ &\Leftrightarrow \quad \sinh(t\pi(1-y)) \ < \ (1-y)\sinh(t\pi) \\ &\Leftrightarrow \quad \sum_{l=0}^{\infty}\frac{(t\pi)^{2l+1}(1-y)^{2l+1}}{(2l+1)!} \ < \ \sum_{l=0}^{\infty}\frac{(t\pi)^{2l+1}(1-y)}{(2l+1)!},\end{aligned}$$

which is valid for positive t. This completes the proof of (35). □

6. Error Splitting. Now, we define a splitting of the discretization error $u_{h_x,h_y} - u$ into three terms $\Gamma^{(1)}_{h_x}$, $\Gamma^{(2)}_{h_y}$, and Γ_{h_x,h_y}. Here, $\Gamma^{(1)}_{h_x}$ and $\Gamma^{(2)}_{h_y}$ are discretization errors of one-dimensional problems, and Γ_{h_x,h_y} involves all solutions u, u_{h_x,h_y}, $u_{h_x,0}$, and u_{0,h_y}. The results from the previous sections concerning these four solutions and the functions μ_k, ν_k, λ_k, and T will enable us to show under which circumstances this splitting is of the type (1). Thus, writing

$$u_{h_x,h_y} - u \quad = \quad \Gamma^{(1)}_{h_x} + \Gamma^{(2)}_{h_y} + \Gamma_{h_x,h_y}, \tag{36}$$

where

$$\begin{aligned}\Gamma^{(1)}_{h_x} \quad &:= \quad u_{h_x,0} - u, \\ \Gamma^{(2)}_{h_y} \quad &:= \quad u_{0,h_y} - u, \\ \Gamma_{h_x,h_y} \quad &:= \quad u_{h_x,h_y} - u_{h_x,0} - u_{0,h_y} + u,\end{aligned} \tag{37}$$

we have to study the requirements on the Dirichlet boundary function g, i.e. on b_k, so that

$$\Gamma^{(1)}_{h_x} \quad = \quad h_x^2 \cdot C_1(x,y,h_x), \tag{38}$$

$$\Gamma^{(2)}_{h_y} \quad = \quad h_y^2 \cdot C_2(x,y,h_y), \tag{39}$$

$$\Gamma_{h_x,h_y} \quad = \quad h_x^2 h_y^2 \cdot C(x,y,h_x,h_y) \tag{40}$$

hold, where $|C_1|$, $|C_2|$, and $|C|$ are bounded from above by some $B(x,y)$.

6.1. $\boldsymbol{\Gamma^{(1)}_{h_x}}$. We start with $\Gamma^{(1)}_{h_x}$. From (18) and (7), we know that

$$\begin{aligned}\Gamma^{(1)}_{h_x} \;=\; \sum_{k=1}^{\infty} b_k \sin(k\pi x)\Big(T(\nu_k,y)-T(k,y)\Big) \;&=\; \sum_{k=1}^{N_x-1} b_k \sin(k\pi x)\Big(T(\nu_k,y)-T(k,y)\Big)\\ &\quad+\; \sum_{k=N_x}^{\infty} b_k \sin(k\pi x)\Big(T(\nu_k,y)-T(k,y)\Big).\end{aligned}$$

First, we deal with the finite sum. Looking at lemmas 6 and 2, we can verify immediately that

$$\Big|T(\nu_k,y)-T(k,y)\Big| \;=\; \Big|T(\xi_k,y)P(\xi_k,y)\,(\nu_k-k)\Big| \;\le\; \mathrm{e}^{-\xi_k\pi y}\cdot \pi y\cdot k^3 h_x^2 |c_1(k,h_x)|$$

for a $\xi_k \in (\nu_k,k)$. Since $0<k<N_x$, (26) shows that $\xi_k > \frac{2}{\pi}k$ and, consequently, that

$$\begin{aligned}\left|\sum_{k=1}^{N_x-1} b_k \sin(k\pi x)\Big(T(\nu_k,y)-T(k,y)\Big)\right| \;&\le\; h_x^2\cdot\frac{\pi^3 y}{24}\cdot\sum_{k=1}^{N_x-1}|b_k|k^3\mathrm{e}^{-2ky}\\ &\le\; h_x^2\cdot\frac{c_g\pi^3 y}{24}\cdot\sum_{k=1}^{\infty}k^{-\varepsilon}\mathrm{e}^{-2ky},\end{aligned}$$

where the series converges for all $0<y<1$.

Next, we turn to the infinite sum in our representation of $\Gamma^{(1)}_{h_x}$. Because of (6) and lemma 6, we get

$$\begin{aligned}\left|\sum_{k=N_x}^{\infty} b_k \sin(k\pi x)\Big(T(\nu_k,y)-T(k,y)\Big)\right| \;\le\; 2\cdot\sum_{k=N_x}^{\infty}|b_k| \;&\le\; 2c_g\cdot\sum_{k=N_x}^{\infty}\frac{1}{k^{3+\varepsilon}}\\ &\le\; h_x^2\cdot 2c_g\cdot\sum_{k=N_x}^{\infty}\frac{1}{k^{1+\varepsilon}}.\end{aligned}$$

Thus, (38) has been proved valid.

6.2. $\boldsymbol{\Gamma^{(2)}_{h_y}}$. Next, we look at $\Gamma^{(2)}_{h_y}$. u_{0,h_y} and u were defined in (23) and (7), respectively:

$$\Gamma^{(2)}_{h_y} \;=\; u_{0,h_y}-u \;=\; \sum_{k=1}^{\infty} b_k\sin(k\pi x)\Big(T(\lambda_k,y)-T(k,y)\Big).$$

Therefore,

$$\left|\Gamma^{(2)}_{h_y}\right| \;\le\; h_y^2\cdot\sum_{k=1}^{\infty}|b_k|\,\frac{T(\lambda_k,y)-T(k,y)}{h_y^2} \;\le\; h_y^2\cdot c_g\cdot\frac{1}{2y^2}\cdot\sum_{k=1}^{\infty}\frac{1}{k^{3+\varepsilon}},$$

if we take into account the following

Lemma 7. *For all $k\in\mathbb{N}$, we have the relation*

$$\frac{T(\lambda_k,y)-T(k,y)}{h_y^2} \;\le\; \frac{\mathrm{e}^{-\lambda_k\pi y}-\mathrm{e}^{-k\pi y}}{h_y^2} \;\le\; \frac{1}{2y^2}.$$

Proof. First, the results of lemma 6 show that $T(t,y) - \mathrm{e}^{-t\pi y}$ increases monotonously in $t > 0$:

$$\frac{\partial}{\partial t}\Big(T(t,y) - \mathrm{e}^{-t\pi y}\Big) \;=\; T(t,y)P(t,y) + \pi y \mathrm{e}^{-t\pi y} \;\geq\; 0,$$

because $T(t,y) \in [0, \mathrm{e}^{-t\pi y}]$ and $P(t,y) \in [-\pi y, 0]$. Since $\lambda_k < k$ for all $k \in \mathbb{N}$, we get

$$\frac{T(\lambda_k, y) - T(k,y)}{h_y^2} \;\leq\; \frac{\mathrm{e}^{-\lambda_k \pi y} - \mathrm{e}^{-k\pi y}}{h_y^2}.$$

With the definition

$$F(t,h) \;:=\; \frac{\operatorname{arsinh}\left(th\frac{\pi}{2}\right)}{h\frac{\pi}{2}}, \tag{41}$$

we get $\lambda_k = F(k, h_y)$ and $\lim_{h_y \to 0} F(k, h_y) = k$. Thus, we can write

$$\begin{aligned}
\frac{\mathrm{e}^{-\lambda_k \pi y} - \mathrm{e}^{-k\pi y}}{h_y^2} &= \frac{\mathrm{e}^{-F(k,h_y)\pi y} - \mathrm{e}^{-F(k,0)\pi y}}{h_y^2} \\
&= \frac{1}{h_y} \cdot \frac{\partial}{\partial h_y} \mathrm{e}^{-F(k,h_y)\pi y}\Big|_{\alpha h_y} \\
&= \frac{1}{h_y} \cdot \pi y \cdot \mathrm{e}^{-F(k,\alpha h_y)\pi y} \cdot \left| \frac{\frac{k\left(\frac{\pi}{2}\right)^2 \alpha h_y}{\sqrt{1 + \left(k\frac{\pi}{2}\alpha h_y\right)^2}} - \frac{\pi}{2}\operatorname{arsinh}\left(k\frac{\pi}{2}\alpha h_y\right)}{\left(\frac{\pi}{2}\right)^2 \alpha^2 h_y^2} \right|
\end{aligned}$$

for some $\alpha \in (0,1)$. Defining $w := \frac{\pi}{2}\alpha h_y k$ and using $\mathrm{e}^{-x} \leq \frac{3!}{x^3}$ for $x > 0$, we finally obtain

$$\begin{aligned}
\frac{\mathrm{e}^{-\lambda_k \pi y} - \mathrm{e}^{-k\pi y}}{h_y^2} &\leq \mathrm{e}^{-F(k,\alpha h_y)\pi y} \cdot \pi y \left(\frac{\pi}{2}\right)^2 \alpha \cdot \left| \frac{\frac{w}{\sqrt{1+w^2}} - \operatorname{arsinh}(w)}{\left(\frac{\pi}{2}\alpha h_y\right)^3} \right| \\
&\leq \frac{6}{\pi^3 y^3} \cdot \pi y \frac{\pi^2}{4} \cdot \left| \frac{\frac{w}{\sqrt{1+w^2}} - \operatorname{arsinh}(w)}{F^3(k, \alpha h_y) \cdot \left(\frac{\pi}{2}\alpha h_y\right)^3} \right| \\
&= \frac{3}{2y^2} \cdot \left| \frac{\frac{w}{\sqrt{1+w^2}} - \operatorname{arsinh}(w)}{\operatorname{arsinh}^3(w)} \right| \\
&\leq \frac{1}{2y^2},
\end{aligned}$$

which completes the proof of lemma 7 and (39). □

6.3. Γ_{h_x,h_y}. Finally, we turn to Γ_{h_x,h_y}. With the definitions from section 3, we get

$$\begin{aligned}\Gamma_{h_x,h_y} &:= u_{h_x,h_y} - u_{h_x,0} - u_{0,h_y} + u \\ &= \sum_{k=1}^{N_x-1} b_k \sin(k\pi x)\Big(T(\mu_k,y) - T(\nu_k,y) - T(\lambda_k,y) + T(k,y)\Big) \\ &+ \sum_{k=N_x}^{\infty} b_k \sin(k\pi x)\Big(T(\mu_k,y) - T(\nu_k,y) - T(\lambda_k,y) + T(k,y)\Big).\end{aligned}$$

Again, let us start with the finite sum. With the function $F(t,h)$ defined in (41) and with

$$S(t,h,y) := T\Big(F(t,h),y\Big) - T(t,y), \tag{42}$$

the relations $\lambda_k = F(k,h_y)$ and $\mu_k = F(\nu_k,h_y)$ (cf. (28) and (31)) lead to

$$\begin{aligned}& T(\mu_k,y) - T(\nu_k,y) - T(\lambda_k,y) + T(k,y) \\ =\; & S(\nu_k,h_y,y) - S(k,h_y,y) \\ =\; & S_t(\xi_k,h_y,y)\cdot(\nu_k - k) \\ =\; & \Big[T_t\Big(F(\xi_k,h_y),y\Big)\cdot F_t(\xi_k,h_y) - T_t(\xi_k,y)\Big]\cdot(\nu_k - k)\end{aligned}$$

for a $\xi_k \in (\nu_k,k) \subseteq \left(\frac{2}{\pi}k,k\right)$, where S_t, T_t, and F_t denote the partial derivatives of S, F, and T with respect to their first argument. Furthermore, since $\lim_{h\to 0} F(t,h) = t$ and $\lim_{h\to 0} F_t(t,h) = 1$, we obtain

$$\begin{aligned}& T_t\Big(F(\xi_k,h_y),y\Big)\cdot F_t(\xi_k,h_y) - T_t(\xi_k,y) \\ =\; & T_t\Big(F(\xi_k,h_y),y\Big)\cdot F_t(\xi_k,h_y) - T_t\Big(F(\xi_k,0),y\Big)\cdot F_t(\xi_k,0) \\ =\; & \frac{\partial}{\partial h_y}\left(T_t\Big(F(\xi_k,h_y),y\Big)\cdot F_t(\xi_k,h_y)\right)\bigg|_{\alpha h_y} \cdot\; h_y\end{aligned}$$

for some $\alpha \in (0,1)$. Together with (34), we receive

$$\begin{aligned}& \frac{\partial}{\partial h_y}\left(T_t\Big(F(\xi_k,h_y),y\Big)\cdot F_t(\xi_k,h_y)\right)\bigg|_{\alpha h_y} \\ =\; & T_{tt}\Big(F(\xi_k,\alpha h_y),y\Big)\cdot F_{h_y}(\xi_k,\alpha h_y)\cdot F_t(\xi_k,\alpha h_y) + T_t\Big(F(\xi_k,\alpha h_y),y\Big)\cdot F_{t,h_y}(\xi_k,\alpha h_y) \\ =\; & T\cdot\left(\Big(P^2 + P_t\Big)\cdot F_{h_y}(\xi_k,\alpha h_y)\cdot F_t(\xi_k,\alpha h_y) + P\cdot F_{t,h_y}(\xi_k,\alpha h_y)\right),\end{aligned}$$

if missing arguments are expanded to $\Big(F(\xi_k,h_y),y\Big)$. Looking at the partial derivatives

$$\begin{aligned}F_{h_y}(\xi_k,\alpha h_y) &= \frac{\frac{\pi}{2}}{\left(\frac{\pi}{2}\alpha h_y\right)^2}\cdot\left(\frac{w}{\sqrt{1+w^2}} - \operatorname{arsinh}(w)\right), \\ F_t(\xi_k,\alpha h_y) &= \frac{1}{\sqrt{1+w^2}}, \quad \text{and} \\ F_{t,h_y}(\xi_k,\alpha h_y) &= \frac{-\xi_k\frac{\pi}{2}\cdot w}{\sqrt{1+w^2}^3},\end{aligned}$$

where $w := \xi_k \frac{\pi}{2} \alpha h_y \in (0,\infty)$, we obtain

$$
\begin{aligned}
& T \cdot \left| \left(P^2 + P_t\right) \cdot F_{h_y}(\xi_k, \alpha h_y) \cdot F_t(\xi_k, \alpha h_y) + P \cdot F_{t,h_y}(\xi_k, \alpha h_y) \right| \\
\leq\ & T \cdot \left(\left|P^2 + P_t\right| \cdot \frac{\frac{\pi}{2}}{\sqrt{1+w^2}} \cdot \left| \frac{\frac{w}{\sqrt{1+w^2}} - \operatorname{arsinh}(w)}{\left(\frac{\pi}{2}\alpha h_y\right)^2} \right| + |P| \cdot \frac{\xi_k \frac{\pi}{2} \cdot w}{\sqrt{1+w^2}^3} \right) \\
=\ & \frac{h_y}{\xi_k} \cdot T \cdot \left(\left|P^2 + P_t\right| \cdot \frac{\left(\frac{\pi}{2}\right)^2 \alpha \cdot w}{\left(\frac{\pi}{2}\alpha h_y\right) \cdot \sqrt{1+w^2}} \cdot \left| \frac{\frac{w}{\sqrt{1+w^2}} - \operatorname{arsinh}(w)}{\left(\frac{\pi}{2}\alpha h_y\right)^3} \right| + \right. \\
& + \left. |P| \cdot \frac{\left(\frac{\pi}{2}\right)^2 \alpha \cdot w^3}{\left(\frac{\pi}{2}\alpha h_y\right)^3 \cdot \sqrt{1+w^2}^3} \right) \\
\leq\ & \frac{h_y}{\xi_k} \cdot \left(\frac{\pi}{2}\right)^2 \cdot \mathrm{e}^{-F(\xi_k,\alpha h_y)\cdot \pi y} \cdot \\
& \left(\left|P^2 + P_t\right| \cdot \frac{w}{\left(\frac{\pi}{2}\alpha h_y\right) \cdot \sqrt{1+w^2}} \cdot \left| \frac{\frac{w}{\sqrt{1+w^2}} - \operatorname{arsinh}(w)}{\left(\frac{\pi}{2}\alpha h_y\right)^3} \right| + |P| \cdot \frac{w^3}{\left(\frac{\pi}{2}\alpha h_y\right)^3 \cdot \sqrt{1+w^2}^3} \right),
\end{aligned}
$$

according to (33). Since $\mathrm{e}^{-x} \leq \frac{m!}{x^m}$ for each $x > 0$ and an arbitrary integer $m > 0$, we get

$$
\begin{aligned}
& T \cdot \left| \left(P^2 + P_t\right) \cdot F_{h_y}(\xi_k, \alpha h_y) \cdot F_t(\xi_k, \alpha h_y) + P \cdot F_{t,h_y}(\xi_k, \alpha h_y) \right| \quad \leq \\
\leq\ & \frac{h_y}{k} \cdot \left(\frac{\pi}{2}\right)^3 \cdot \left(\frac{4!}{\pi^4 y^4} \left|P^2 + P_t\right| \cdot \frac{w}{\left(F(\xi_k,\alpha h_y)\alpha h_y \frac{\pi}{2}\right) \cdot \sqrt{1+w^2}} \cdot \left| \frac{\frac{w}{\sqrt{1+w^2}} - \operatorname{arsinh}(w)}{\left(F(\xi_k,\alpha h_y)\frac{\pi}{2}\alpha h_y\right)^3} \right| \right. \\
& + \left. \frac{3!}{\pi^3 y^3} |P| \cdot \frac{w^3}{\left(F(\xi_k,\alpha h_y)\frac{\pi}{2}\alpha h_y\right)^3 \sqrt{1+w^2}^3} \right) \\
=\ & \frac{h_y}{k} \cdot \frac{\pi^3}{8} \cdot \left(\frac{24}{\pi^4 y^4} \left|P^2 + P_t\right| \cdot \frac{w}{\operatorname{arsinh}(w) \cdot \sqrt{1+w^2}} \cdot \left| \frac{\frac{w}{\sqrt{1+w^2}} - \operatorname{arsinh}(w)}{\operatorname{arsinh}^3(w)} \right| + \right. \\
& + \left. \frac{6}{\pi^3 y^3} |P| \cdot \frac{w^3}{\operatorname{arsinh}^3(w) \cdot \sqrt{1+w^2}^3} \right) \\
\leq\ & \frac{h_y}{k} \cdot \frac{\pi^3}{8} \cdot \left(\frac{8}{\pi^4 y^4} \left|P^2 + P_t\right| \ + \ \frac{6}{\pi^3 y^3} |P| \right),
\end{aligned}
$$

because

$$
0 \leq \frac{w}{\operatorname{arsinh}(w)\sqrt{1+w^2}} \leq 1, \quad 0 \leq \left| \frac{\frac{w}{\sqrt{1+w^2}} - \operatorname{arsinh}(w)}{\operatorname{arsinh}^3(w)} \right| \leq \frac{1}{3}.
$$

From (35), we know that $|P(t,y)| \leq \pi y$. Analogously, we can show that $|P_t(t,y)| \leq \frac{\pi^2}{3}(2y - y^2)$. These relations, finally, lead us to

$$
T \cdot \left| \left(P^2 + P_t\right) \cdot F_{h_y}(\xi_k, \alpha h_y) \cdot F_t(\xi_k, \alpha h_y) + P \cdot F_{t,h_y}(\xi_k, \alpha h_y) \right| \ \leq \ \frac{h_y \cdot 3\pi}{k \cdot y^3}
$$

and, consequently, to

$$
\begin{aligned}
&\left|\sum_{k=1}^{N_x-1} b_k \sin(k\pi x)\Big(T(\mu_k,y)-T(\nu_k,y)-T(\lambda_k,y)+T(k,y)\Big)\right| \\
\leq\;& \sum_{k=1}^{N_x-1} |b_k|\cdot\left|T_t\Big(F(\xi_k,h_y),y\Big)\cdot F_t(\xi_k,h_y)-T_t(\xi_k,y)\right|\cdot|\nu_k-k| \\
\leq\;& \sum_{k=1}^{N_x-1} |b_k|\cdot h_y\cdot\frac{h_y\cdot 3\pi}{k\cdot y^3}\cdot h_x^2 k^3|c_1(k,h_x)| \\
\leq\;& h_x^2h_y^2\cdot\frac{\pi^3}{8y^3}\cdot c_g\cdot\sum_{k=1}^{\infty}\frac{1}{k^{1+\varepsilon}},
\end{aligned}
$$

because $|b_k|\leq\frac{c_g}{k^{3+\varepsilon}}$ (see (6)).

Finally, we turn to the infinite sum in our representation of Γ_{h_x,h_y} in order to complete the proof of (40):

$$
\begin{aligned}
&\left|\sum_{k=N_x}^{\infty} b_k \sin(k\pi x)\Big(T(\mu_k,y)-T(\nu_k,y)-T(\lambda_k,y)+T(k,y)\Big)\right| \\
\leq\;& h_y^2\;\cdot\sum_{k=N_x}^{\infty}|b_k|\cdot\left|\frac{T(\mu_k,y)-T(\nu_k,y)}{h_y^2}-\frac{T(\lambda_k,y)-T(k,y)}{h_y^2}\right| \\
\leq\;& h_x^2h_y^2\cdot c_g\cdot\sum_{k=N_x}^{\infty}\frac{1}{k^{1+\varepsilon}}\cdot\left(\left|\frac{T(\mu_k,y)-T(\nu_k,y)}{h_y^2}\right|+\left|\frac{T(\lambda_k,y)-T(k,y)}{h_y^2}\right|\right).
\end{aligned}
$$

In section 6.2, we have already shown that

$$
\left|\frac{T(\lambda_k,y)-T(k,y)}{h_y^2}\right| \;=\; \frac{T(\lambda_k,y)-T(k,y)}{h_y^2} \;\leq\; \frac{1}{2y^2}
$$

for all k and h_y. Now, we turn to $\Big(T(\mu_k,y)-T(\nu_k,y)\Big)/h_y^2$. From (25) and (31),

$$
\nu_k \;=\; \frac{\sin\left(\frac{k\pi}{2N_x}\right)}{\frac{\pi}{2N_x}} \;=\; \frac{\sinh\left(\frac{\mu_k\pi h_y}{2}\right)}{\frac{\pi h_y}{2}},
$$

we can see the $4N_x$-periodic behaviour of ν_k and, hence, of μ_k, and the periodic behaviour of the relationship between ν_k and μ_k (see the proof of lemma 5):

$$
k\neq l\cdot 2N_x,\; l\in\mathbb{Z} \;\Rightarrow\; 0<|\mu_k|<|\nu_k|\,.
$$

Only for $k=l\cdot 2N_x$, $l\in\mathbb{Z}$, we get $\nu_k=\mu_k=0$ and, consequently, $T(\mu_k,y)=T(\nu_k,y)$. Thus, without loss of generality, let $\nu_k\neq 0$. Then, due to the proof of lemma 7,

$$
\left|\frac{T(\mu_k,y)-T(\nu_k,y)}{h_y^2}\right| \;=\; \frac{T(|\mu_k|,y)-T(|\nu_k|,y)}{h_y^2} \;\leq\; \frac{\mathrm{e}^{-|\mu_k|\pi y}-\mathrm{e}^{-|\nu_k|\pi y}}{h_y^2}.
$$

The rest of the proof follows exactly the lines of section 6.2, just with $|\nu_k|$ instead of k, leading to

$$\frac{T(\mu_k,y)-T(\nu_k,y)}{h_y^2} \;\le\; \frac{1}{2y^2}.$$

This completes the proof of (40) and, hence, of the error splitting (36) and (37). Consequently, our smoothness requirement (6) on the Dirichlet boundary function g is sufficient for the existence of a splitting of $u_{h_x,h_y}(x,y)-u(x,y)$ according to (1). Summing up our results of this section, we get the following

THEOREM 6. *If the Fourier coefficients b_k of $\tilde{g}$, the 2-periodic and 0-symmetric extension of the Dirichlet boundary function g, satisfy $|b_k| \le \frac{c_g}{k^{3+\varepsilon}}$, the error $u_{h_x,h_y}(x,y)-u(x,y)$ on the grid $G_{i,j}$ with meshwidths $h_x = 2^{-i}$ and $h_y = 2^{-j}$ can be written as*

$$u_{h_x,h_y}(x,y)-u(x,y) = C_1(x,y,h_x)h_x^2 + C_2(x,y,h_y)h_y^2 + C(x,y,h_x,h_y)h_x^2h_y^2, \tag{43}$$

where $|C_1|$, $|C_2|$, and $|C|$ are bounded from above by some $B(x,y)$ independent of h_x and h_y.

Note that, actually, the condition concerning the Fourier coefficients b_k is a smoothness requirement on the Dirichlet boundary function g. If, e.g., $g \in C^4([0,1])$ with $g(0) = g(1) = g''(0) = g''(1) = 0$, it is a well-known fact from Fourier theory that $b_k \le \frac{c_g}{k^4}$. In the more general case, an arbitrary continuous boundary function that is four times partially differentiable on each edge of $\bar{\Omega}$ can be reduced to the situation of our model problem (4), (5). For that, we have to add (harmonic) functions of the type $\cos(\frac{\pi}{2}x)\sinh(\pi(1-y))$ and $\cos(\frac{\pi}{2}(1-x))\sinh(\pi(1-y))$ on each edge in order to force the second partial derivatives to zero at the corners. Afterwards, the (harmonic) bilinear interpolant of the values at the four corners of $\bar{\Omega}$ has to be subtracted, and, finally, superposition enables us to consider only functions vanishing on three edges of $\bar{\Omega}$. Here, it is important that all those transformations do not influence the convergence behaviour. Therefore, it is sufficient to require $g \in C^4([0,1])$ in order to obtain $b_k \le \frac{c_g}{k^4}$.

Finally, combining theorems 1 and 6, we get the crucial result of this paper: If the Dirichlet boundary function g fulfills the smoothness requirement of theorem 6, then, according to theorem 1, the error $e_n^c(x,y)$ of the combined solution $u_n^c(x,y)$ is of the order $O(h_n^2 \log_2(h_n^{-1}))$, i.e., apart from the logarithmic factor, the sparse grid solution $u_n^c(x,y)$ is of the same accuracy as the full grid solution u_{h_n,h_n}.

7. Numerical Experiments. In the following, we study the convergence behaviour of the combination technique in the case of Laplace's equation on $\Omega =]0,1[^2$ with Dirichlet boundary conditions and the solutions

$$u^{(\alpha)}(x,y) \;:=\; Im\left((x-\frac{1}{2}+iy)^\alpha\right) \;=\; r^\alpha(x,y)\cdot\sin\Big(\alpha\cdot\varphi(x,y)\Big), \quad \alpha > 0,$$

where $r(x,y) = \sqrt{(x-\frac{1}{2})^2+y^2}$ and $\varphi(x,y) = \arg(x-\frac{1}{2}+iy)$. Here, the parameter α indicates the smoothness of $u^{(\alpha)}$ in the critical point $(\frac{1}{2},0)$. If, e.g., $\alpha \ge 4$, then we have $g(x) := u^{(\alpha)}(x,0) \in C^4([0,1])$. Figure 2 shows $u^{(\frac{1}{2})}$ with its singularity in $(\frac{1}{2},0)$.

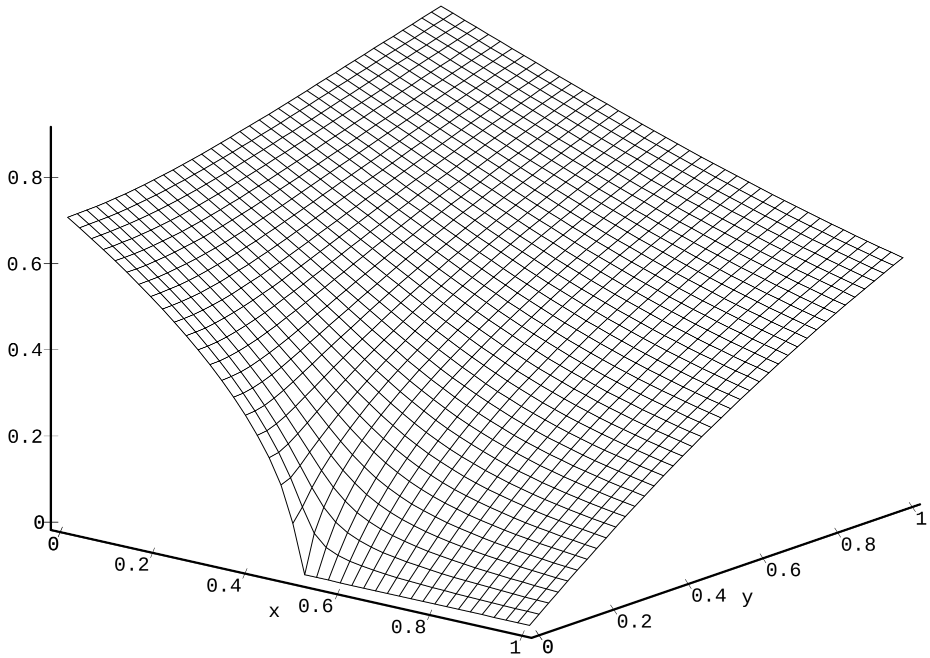

FIG. 2. $u^{(\frac{1}{2})}$.

All of the following computations were performed on a HP 9000/755 workstation with 128 MB main storage. The code has been written in FORTRAN. For the discretization on the full grids $G_{i,j}$, we used finite differences as indicated in (9). The resulting systems were solved with the help of the NAG library routine D03EDF written by P. Wesseling.

As introduced before, $e_n^c(x,y) := u_n^c(x,y) - u(x,y)$ denotes the error of the combined solution. Concerning the pointwise error, we consider the two points $P_1 := (\frac{1}{2}, \frac{1}{2})$ and $P_2 := (\frac{1}{4}, \frac{1}{4})$. The quotients $\rho_{P_i} := \left| e_{n-1}^c(P_i)/e_n^c(P_i) \right|$ show the decrease in the pointwise error when proceeding from level $n-1$ to n. For $n = 2l-1$ and $n = 2l$ ($l \in \mathbb{N}$), the discrete maximum- and L_2-norm are both computed on the full grid $G_{l,l}$, the finest full grid contained in $\tilde{G}_{n,n}$. Therefore, we always compare the norms on levels $n-2$ and n and define $\rho_\infty := \|e_{n-2}^c\|_\infty / \|e_n^c\|_\infty$ and $\rho_2 := \|e_{n-2}^c\|_2 / \|e_n^c\|_2$.

In table 1, we present the results for $\alpha = 0.5$. From **[12]**, we know that for full grids $G_{n,n}$, a pointwise convergence rate of $O(h_n^{3/2})$ ($h_n = 2^{-n}$) can be expected in a sufficient distance from the singularity. Now, for the combined solutions on sparse grids, we observe values of ρ_{P_1} and and ρ_{P_2} close to $2^{3/2} = \sqrt{8}$, too. Looking at P_2, we can see that the combination technique seems to cause an additional logarithmic factor like in the smooth case. Thus, the combined solution achieves nearly full grid accuracy, but involves significantly less grid points. Concerning the norms, we can observe a behaviour comparable to full grids. Proceeding from $n-2$ to n leads to the same convergence rates ρ_2 of $2^{3/2} = \sqrt{8}$ for the L_2-norm and ρ_∞ of $2^{1/2} = \sqrt{2}$ for the maximum-norm as we get for full grids when proceeding from $n-1$ to n (see **[12]** and **[9]**). Since, in both

TABLE 1
$\alpha = 0.5$

n	$e_n^c(P_1)$	ρ_{P_1}	$e_n^c(P_2)$	ρ_{P_2}	$\|\|e_n^c\|\|_\infty$	ρ_∞	$\|\|e_n^c\|\|_2$	ρ_2
1	$-4.8552 \cdot 10^{-02}$		—		$4.8552 \cdot 10^{-02}$		$4.8552 \cdot 10^{-02}$	
2	$-1.9259 \cdot 10^{-02}$	2.521	—	—	$1.9259 \cdot 10^{-02}$	—	$1.9259 \cdot 10^{-02}$	—
3	$-4.7039 \cdot 10^{-03}$	4.094	$-4.6221 \cdot 10^{-03}$	—	$3.5937 \cdot 10^{-02}$	1.351	$1.2571 \cdot 10^{-02}$	3.862
4	$-1.7749 \cdot 10^{-04}$	26.502	$-2.4920 \cdot 10^{-03}$	1.855	$1.3863 \cdot 10^{-02}$	1.389	$4.9091 \cdot 10^{-03}$	3.923
5	$4.0667 \cdot 10^{-04}$	0.436	$-1.1839 \cdot 10^{-03}$	2.105	$2.5689 \cdot 10^{-02}$	1.399	$3.9633 \cdot 10^{-03}$	3.172
6	$2.2504 \cdot 10^{-04}$	1.807	$-5.1783 \cdot 10^{-04}$	2.286	$9.9078 \cdot 10^{-03}$	1.399	$1.5510 \cdot 10^{-03}$	3.165
7	$8.9342 \cdot 10^{-05}$	2.519	$-2.1420 \cdot 10^{-04}$	2.418	$1.8199 \cdot 10^{-02}$	1.412	$1.3215 \cdot 10^{-03}$	2.999
8	$3.3083 \cdot 10^{-05}$	2.701	$-8.5143 \cdot 10^{-05}$	2.516	$7.0177 \cdot 10^{-03}$	1.412	$5.1653 \cdot 10^{-04}$	3.003
9	$1.1989 \cdot 10^{-05}$	2.759	$-3.2867 \cdot 10^{-05}$	2.591	$1.2871 \cdot 10^{-02}$	1.414	$4.5301 \cdot 10^{-04}$	2.917
10	$4.2925 \cdot 10^{-06}$	2.793	$-1.2414 \cdot 10^{-05}$	2.648	$4.9633 \cdot 10^{-03}$	1.414	$1.7701 \cdot 10^{-04}$	2.918
11	$1.5260 \cdot 10^{-06}$	2.813	$-4.6134 \cdot 10^{-06}$	2.691	$9.1015 \cdot 10^{-03}$	1.414	$1.5767 \cdot 10^{-04}$	2.873
12	$5.4037 \cdot 10^{-07}$	2.824	$-1.6937 \cdot 10^{-06}$	2.724	$3.5097 \cdot 10^{-03}$	1.414	$6.1605 \cdot 10^{-05}$	2.873
13	$1.9095 \cdot 10^{-07}$	2.830	$-6.1608 \cdot 10^{-07}$	2.749	$6.4357 \cdot 10^{-03}$	1.414	$5.5307 \cdot 10^{-05}$	2.851
14	$6.7406 \cdot 10^{-08}$	2.833	$-2.2255 \cdot 10^{-07}$	2.768	$2.4817 \cdot 10^{-03}$	1.414	$2.1610 \cdot 10^{-05}$	2.851
15	$2.3787 \cdot 10^{-08}$	2.834	$-7.9972 \cdot 10^{-08}$	2.783	$4.5508 \cdot 10^{-03}$	1.414	$1.9478 \cdot 10^{-05}$	2.839
16	$8.3938 \cdot 10^{-09}$	2.834	$-2.8623 \cdot 10^{-08}$	2.794	$1.7548 \cdot 10^{-03}$	1.414	$7.6103 \cdot 10^{-06}$	2.840

cases, the number of grid points increases approximately by a factor of 4 (the influence of the logarithmic term being neglected), the full grid solution u_{h_n,h_n} and the combined solution u_{2n}^c are of the same quality.

TABLE 2
$\alpha = 0.9$

n	$e_n^c(P_1)$	ρ_{P_1}	$e_n^c(P_2)$	ρ_{P_2}	$\|\|e_n^c\|\|_\infty$	ρ_∞	$\|\|e_n^c\|\|_2$	ρ_2
1	$-7.4688 \cdot 10^{-03}$		—		$7.4688 \cdot 10^{-03}$		$7.4688 \cdot 10^{-03}$	
2	$-2.5289 \cdot 10^{-03}$	2.953	—	—	$2.5289 \cdot 10^{-03}$	—	$2.5289 \cdot 10^{-03}$	—
3	$-5.3009 \cdot 10^{-04}$	4.771	$-3.7809 \cdot 10^{-04}$	—	$3.9451 \cdot 10^{-03}$	1.893	$1.3566 \cdot 10^{-03}$	5.506
4	$-1.4235 \cdot 10^{-05}$	37.238	$-1.4551 \cdot 10^{-04}$	2.598	$1.2745 \cdot 10^{-03}$	1.984	$4.3797 \cdot 10^{-04}$	5.774
5	$3.7429 \cdot 10^{-05}$	0.380	$-5.2373 \cdot 10^{-05}$	2.778	$2.0998 \cdot 10^{-03}$	1.879	$3.1188 \cdot 10^{-04}$	4.350
6	$1.8191 \cdot 10^{-05}$	2.058	$-1.7967 \cdot 10^{-05}$	2.915	$6.7916 \cdot 10^{-04}$	1.877	$1.0103 \cdot 10^{-04}$	4.335
7	$6.3389 \cdot 10^{-06}$	2.870	$-5.9319 \cdot 10^{-06}$	3.029	$1.1252 \cdot 10^{-03}$	1.866	$7.8141 \cdot 10^{-05}$	3.991
8	$2.0267 \cdot 10^{-06}$	3.128	$-1.8981 \cdot 10^{-06}$	3.125	$3.6396 \cdot 10^{-04}$	1.866	$2.5313 \cdot 10^{-05}$	3.991
9	$6.2366 \cdot 10^{-07}$	3.250	$-5.9230 \cdot 10^{-07}$	3.205	$6.0299 \cdot 10^{-04}$	1.866	$2.0270 \cdot 10^{-05}$	3.855
10	$1.8689 \cdot 10^{-07}$	3.337	$-1.8116 \cdot 10^{-07}$	3.269	$1.9505 \cdot 10^{-04}$	1.866	$6.5657 \cdot 10^{-06}$	3.855
11	$5.4910 \cdot 10^{-08}$	3.404	$-5.4526 \cdot 10^{-08}$	3.322	$3.2314 \cdot 10^{-04}$	1.866	$5.3451 \cdot 10^{-06}$	3.792
12	$1.5892 \cdot 10^{-08}$	3.455	$-1.6198 \cdot 10^{-08}$	3.366	$1.0452 \cdot 10^{-04}$	1.866	$1.7314 \cdot 10^{-06}$	3.792
13	$4.5456 \cdot 10^{-09}$	3.496	$-4.7602 \cdot 10^{-09}$	3.403	$1.7317 \cdot 10^{-04}$	1.866	$1.4209 \cdot 10^{-06}$	3.762
14	$1.2879 \cdot 10^{-09}$	3.529	$-1.3862 \cdot 10^{-09}$	3.434	$5.6013 \cdot 10^{-05}$	1.866	$4.6026 \cdot 10^{-07}$	3.762
15	$3.6213 \cdot 10^{-10}$	3.556	$-4.0057 \cdot 10^{-10}$	3.461	$9.2797 \cdot 10^{-05}$	1.866	$3.7924 \cdot 10^{-07}$	3.747
16	$1.0117 \cdot 10^{-10}$	3.579	$-1.1497 \cdot 10^{-10}$	3.484	$3.0017 \cdot 10^{-05}$	1.866	$1.2284 \cdot 10^{-07}$	3.747

The tables 2–5 show some results for increasing values of α. In the case of $\alpha = 0.9$, we again can observe convergence rates ρ_{P_1}, ρ_{P_2}, ρ_∞, and ρ_2 that are only slightly worse than the corresponding full grid values $2^{0.9} \approx 1.866$ and $2^{1.9} \approx 3.732$, respectively, which is probably due to the logarithmic factor. Like for $\alpha = 0.5$, this means a significant gain in comparison with full grids regarding the pointwise rates, but no improvement if norms are of interest.

This situation changes if we look at $\alpha \in \{1.5, 2.5, 3.5\}$. Here, the pointwise errors seem to decrease like $O(h_n^2 \log_2(h_n^{-1}))$. The convergence rates ρ_∞ and ρ_2 show a

Table 3
$\alpha = 1.5$

n	$e_n^c(P_1)$	ρ_{P_1}	$e_n^c(P_2)$	ρ_{P_2}	$\|\|e_n^c\|\|_\infty$	ρ_∞	$\|\|e_n^c\|\|_2$	ρ_2
1	$7.2260 \cdot 10^{-03}$		—		$7.2260 \cdot 10^{-03}$		$7.2260 \cdot 10^{-03}$	
2	$2.2000 \cdot 10^{-03}$	3.285	—	—	$2.2000 \cdot 10^{-03}$	—	$2.2000 \cdot 10^{-03}$	—
3	$5.1219 \cdot 10^{-04}$	4.295	$1.1748 \cdot 10^{-03}$	—	$2.4573 \cdot 10^{-03}$	2.941	$9.7681 \cdot 10^{-04}$	7.398
4	$8.7919 \cdot 10^{-05}$	5.826	$3.6735 \cdot 10^{-04}$	3.198	$6.9889 \cdot 10^{-04}$	3.148	$2.8435 \cdot 10^{-04}$	7.737
5	$1.0937 \cdot 10^{-05}$	8.039	$1.0996 \cdot 10^{-04}$	3.341	$8.4134 \cdot 10^{-04}$	2.921	$1.5243 \cdot 10^{-04}$	6.408
6	$9.9056 \cdot 10^{-07}$	11.041	$3.2374 \cdot 10^{-05}$	3.397	$2.3941 \cdot 10^{-04}$	2.919	$4.4250 \cdot 10^{-05}$	6.426
7	$2.2457 \cdot 10^{-10}$	4410.919	$9.3957 \cdot 10^{-06}$	3.446	$2.9634 \cdot 10^{-04}$	2.839	$2.5350 \cdot 10^{-05}$	6.013
8	$-4.8418 \cdot 10^{-08}$	0.005	$2.6848 \cdot 10^{-06}$	3.500	$8.4386 \cdot 10^{-05}$	2.837	$7.3529 \cdot 10^{-06}$	6.018
9	$-2.3117 \cdot 10^{-08}$	2.094	$7.5565 \cdot 10^{-07}$	3.553	$1.0474 \cdot 10^{-04}$	2.829	$4.3444 \cdot 10^{-06}$	5.835
10	$-8.3321 \cdot 10^{-09}$	2.774	$2.0986 \cdot 10^{-07}$	3.601	$2.9828 \cdot 10^{-05}$	2.829	$1.2597 \cdot 10^{-06}$	5.837
11	$-2.6829 \cdot 10^{-09}$	3.106	$5.7643 \cdot 10^{-08}$	3.641	$3.7030 \cdot 10^{-05}$	2.829	$7.5605 \cdot 10^{-07}$	5.746
12	$-8.1369 \cdot 10^{-10}$	3.297	$1.5691 \cdot 10^{-08}$	3.674	$1.0546 \cdot 10^{-05}$	2.828	$2.1920 \cdot 10^{-07}$	5.747
13	$-2.3790 \cdot 10^{-10}$	3.420	$4.2402 \cdot 10^{-09}$	3.701	$1.3092 \cdot 10^{-05}$	2.828	$1.3261 \cdot 10^{-07}$	5.701
14	$-6.7868 \cdot 10^{-11}$	3.505	$1.1389 \cdot 10^{-09}$	3.723	$3.7284 \cdot 10^{-06}$	2.829	$3.8446 \cdot 10^{-08}$	5.702
15	$-1.9025 \cdot 10^{-11}$	3.567	$3.0432 \cdot 10^{-10}$	3.742	$4.6288 \cdot 10^{-06}$	2.828	$2.3350 \cdot 10^{-08}$	5.679
16	$-5.2638 \cdot 10^{-12}$	3.614	$8.0966 \cdot 10^{-11}$	3.759	$1.3182 \cdot 10^{-06}$	2.828	$6.7698 \cdot 10^{-09}$	5.679

behaviour that is quite comparable to the results one gets using Richardson extrapolation. In fact, numerical experiments in **[14]** indicate the same rates for the combined solution u_{2n}^c and the extrapolated solution $u_n^R := \frac{1}{3}\left(4u_{h_n/2,h_n/2} - u_{h_n,h_n}\right)$. Note that, except for a logarithmic factor, computing u_{2n}^c costs the same as the computation of u_n^R does.

Table 4
$\alpha = 2.5$

n	$e_n^c(P_1)$	ρ_{P_1}	$e_n^c(P_2)$	ρ_{P_2}	$\|\|e_n^c\|\|_\infty$	ρ_∞	$\|\|e_n^c\|\|_2$	ρ_2
1	$5.1095 \cdot 10^{-03}$		—		$5.1095 \cdot 10^{-03}$		$5.1095 \cdot 10^{-03}$	
2	$1.5775 \cdot 10^{-03}$	3.239	—	—	$1.5775 \cdot 10^{-03}$	—	$1.5775 \cdot 10^{-03}$	—
3	$4.5941 \cdot 10^{-04}$	3.434	$-1.2014 \cdot 10^{-05}$	—	$1.0051 \cdot 10^{-03}$	5.084	$5.6306 \cdot 10^{-04}$	9.075
4	$1.2927 \cdot 10^{-04}$	3.554	$-2.0791 \cdot 10^{-05}$	0.578	$2.9221 \cdot 10^{-04}$	5.399	$1.6407 \cdot 10^{-04}$	9.615
5	$3.5944 \cdot 10^{-05}$	3.596	$-1.0301 \cdot 10^{-05}$	2.018	$1.8969 \cdot 10^{-04}$	5.299	$5.9037 \cdot 10^{-05}$	9.537
6	$9.9555 \cdot 10^{-06}$	3.610	$-4.0176 \cdot 10^{-06}$	2.564	$5.4353 \cdot 10^{-05}$	5.376	$1.6558 \cdot 10^{-05}$	9.909
7	$2.7393 \cdot 10^{-06}$	3.634	$-1.3937 \cdot 10^{-06}$	2.883	$3.3093 \cdot 10^{-05}$	5.732	$5.8185 \cdot 10^{-06}$	10.146
8	$7.4777 \cdot 10^{-07}$	3.663	$-4.4966 \cdot 10^{-07}$	3.099	$9.4489 \cdot 10^{-06}$	5.752	$1.5981 \cdot 10^{-06}$	10.361
9	$2.0268 \cdot 10^{-07}$	3.689	$-1.3818 \cdot 10^{-07}$	3.254	$5.8241 \cdot 10^{-06}$	5.682	$5.5103 \cdot 10^{-07}$	10.559
10	$5.4603 \cdot 10^{-08}$	3.712	$-4.1033 \cdot 10^{-08}$	3.368	$1.6627 \cdot 10^{-06}$	5.683	$1.4963 \cdot 10^{-07}$	10.680
11	$1.4624 \cdot 10^{-08}$	3.734	$-1.1885 \cdot 10^{-08}$	3.453	$1.0286 \cdot 10^{-06}$	5.662	$5.0887 \cdot 10^{-08}$	10.829
12	$3.9043 \cdot 10^{-09}$	3.746	$-3.3781 \cdot 10^{-09}$	3.518	$2.9365 \cdot 10^{-07}$	5.662	$1.3730 \cdot 10^{-08}$	10.898
13	$1.0375 \cdot 10^{-09}$	3.763	$-9.4630 \cdot 10^{-10}$	3.570	$1.8180 \cdot 10^{-07}$	5.658	$4.6239 \cdot 10^{-09}$	11.005
14	$2.7475 \cdot 10^{-10}$	3.776	$-2.6202 \cdot 10^{-10}$	3.612	$5.1902 \cdot 10^{-08}$	5.658	$1.2429 \cdot 10^{-09}$	11.047
15	$7.2529 \cdot 10^{-11}$	3.788	$-7.1866 \cdot 10^{-11}$	3.646	$3.2138 \cdot 10^{-08}$	5.657	$4.1578 \cdot 10^{-10}$	11.121
16	$1.9092 \cdot 10^{-11}$	3.799	$-1.9557 \cdot 10^{-11}$	3.675	$9.1748 \cdot 10^{-09}$	5.657	$1.1151 \cdot 10^{-10}$	10.798

As a last example, we want to look at the Dirichlet problem with the solution

$$u(x, y) := \sin(\pi y) \cdot \frac{\sinh(\pi(1-x))}{\sinh(\pi)}.$$

Note that u vanishes on three edges of the unit square. For $x = 0$, we have $g(y) := u(0, y) = \sin(\pi y)$ and, especially, $b_1 = 1$ and $b_k = 0$ for all $k > 1$. Therefore, g perfectly

TABLE 5
$\alpha = 3.5$

n	$e_n^c(P_1)$	ρ_{P_1}	$e_n^c(P_2)$	ρ_{P_2}	$\|\|e_n^c\|\|_\infty$	ρ_∞	$\|\|e_n^c\|\|_2$	ρ_2
1	$-1.7166 \cdot 10^{-02}$		—		$1.7166 \cdot 10^{-02}$		$1.7166 \cdot 10^{-02}$	
2	$-5.5140 \cdot 10^{-03}$	3.113	—	—	$5.5140 \cdot 10^{-03}$	—	$5.5140 \cdot 10^{-03}$	—
3	$-1.7080 \cdot 10^{-03}$	3.228	$-2.4374 \cdot 10^{-03}$	—	$2.4374 \cdot 10^{-03}$	7.043	$1.7699 \cdot 10^{-03}$	9.699
4	$-5.1273 \cdot 10^{-04}$	3.331	$-7.1068 \cdot 10^{-04}$	3.430	$7.1068 \cdot 10^{-04}$	7.759	$5.1468 \cdot 10^{-04}$	10.713
5	$-1.5001 \cdot 10^{-04}$	3.418	$-2.0325 \cdot 10^{-04}$	3.497	$2.7319 \cdot 10^{-04}$	8.922	$1.6012 \cdot 10^{-04}$	11.054
6	$-4.3006 \cdot 10^{-05}$	3.488	$-5.7314 \cdot 10^{-05}$	3.546	$7.4642 \cdot 10^{-05}$	9.521	$4.4061 \cdot 10^{-05}$	11.681
7	$-1.2132 \cdot 10^{-05}$	3.545	$-1.5971 \cdot 10^{-05}$	3.589	$2.5915 \cdot 10^{-05}$	10.542	$1.3333 \cdot 10^{-05}$	12.009
8	$-3.3783 \cdot 10^{-06}$	3.591	$-4.4052 \cdot 10^{-06}$	3.625	$6.9060 \cdot 10^{-06}$	10.808	$3.5685 \cdot 10^{-06}$	12.347
9	$-9.3097 \cdot 10^{-07}$	3.629	$-1.2046 \cdot 10^{-06}$	3.657	$2.2259 \cdot 10^{-06}$	11.642	$1.0535 \cdot 10^{-06}$	12.656
10	$-2.5434 \cdot 10^{-07}$	3.660	$-3.2701 \cdot 10^{-07}$	3.684	$5.8216 \cdot 10^{-07}$	11.863	$2.7759 \cdot 10^{-07}$	12.855
11	$-6.8985 \cdot 10^{-08}$	3.687	$-8.8219 \cdot 10^{-08}$	3.707	$1.8835 \cdot 10^{-07}$	11.818	$8.0189 \cdot 10^{-08}$	13.138
12	$-1.8596 \cdot 10^{-08}$	3.710	$-2.3671 \cdot 10^{-08}$	3.727	$4.8946 \cdot 10^{-08}$	11.894	$2.0921 \cdot 10^{-08}$	13.268
13	$-4.9865 \cdot 10^{-09}$	3.729	$-6.3219 \cdot 10^{-09}$	3.744	$1.6303 \cdot 10^{-08}$	11.553	$5.9337 \cdot 10^{-09}$	13.514
14	$-1.3310 \cdot 10^{-09}$	3.746	$-1.6815 \cdot 10^{-09}$	3.760	$4.2279 \cdot 10^{-09}$	11.577	$1.5376 \cdot 10^{-09}$	13.606
15	$-3.5384 \cdot 10^{-10}$	3.762	$-4.4563 \cdot 10^{-10}$	3.773	$1.4288 \cdot 10^{-09}$	11.410	$4.2951 \cdot 10^{-10}$	13.815
16	$-9.3734 \cdot 10^{-11}$	3.775	$-1.1772 \cdot 10^{-10}$	3.786	$3.7029 \cdot 10^{-10}$	11.418	$1.1075 \cdot 10^{-10}$	13.884

fulfills the smoothness requirement (6). In fact, we learn from table 6 that both the pointwise error and the error norms decrease like $O(h_n^2 \log_2(h_n^{-1}))$. Remember that ρ_∞ and ρ_2 result from comparing levels $n-2$ and n.

TABLE 6
$\sin(\pi y)\sinh(\pi(1-x))/\sinh(\pi)$

n	$e_n^c(P_1)$	ρ_{P_1}	$e_n^c(P_2)$	ρ_{P_2}	$\|\|e_n^c\|\|_\infty$	ρ_∞	$\|\|e_n^c\|\|_2$	ρ_2
1	$5.0732 \cdot 10^{-02}$		—		$5.0732 \cdot 10^{-02}$		$5.0732 \cdot 10^{-02}$	
2	$1.6274 \cdot 10^{-02}$	3.117	—	—	$1.6274 \cdot 10^{-02}$	—	$1.6274 \cdot 10^{-02}$	—
3	$5.0475 \cdot 10^{-03}$	3.224	$7.6309 \cdot 10^{-03}$	—	$9.3968 \cdot 10^{-03}$	5.399	$5.6525 \cdot 10^{-03}$	8.975
4	$1.5164 \cdot 10^{-03}$	3.329	$2.1985 \cdot 10^{-03}$	3.471	$2.7439 \cdot 10^{-03}$	5.931	$1.6439 \cdot 10^{-03}$	9.900
5	$4.4371 \cdot 10^{-04}$	3.418	$6.2177 \cdot 10^{-04}$	3.536	$1.0765 \cdot 10^{-03}$	8.729	$5.2966 \cdot 10^{-04}$	10.672
6	$1.2717 \cdot 10^{-04}$	3.489	$1.7349 \cdot 10^{-04}$	3.584	$2.8984 \cdot 10^{-04}$	9.467	$1.4527 \cdot 10^{-04}$	11.316
7	$3.5858 \cdot 10^{-05}$	3.546	$4.7887 \cdot 10^{-05}$	3.623	$1.0661 \cdot 10^{-04}$	10.098	$4.4719 \cdot 10^{-05}$	11.844
8	$9.9818 \cdot 10^{-06}$	3.592	$1.3101 \cdot 10^{-05}$	3.655	$2.8205 \cdot 10^{-05}$	10.276	$1.1920 \cdot 10^{-05}$	12.187
9	$2.7498 \cdot 10^{-06}$	3.630	$3.5576 \cdot 10^{-06}$	3.683	$8.8161 \cdot 10^{-06}$	12.093	$3.5591 \cdot 10^{-06}$	12.565
10	$7.5105 \cdot 10^{-07}$	3.661	$9.6000 \cdot 10^{-07}$	3.706	$2.2943 \cdot 10^{-06}$	12.294	$9.3415 \cdot 10^{-07}$	12.760
11	$2.0366 \cdot 10^{-07}$	3.688	$2.5765 \cdot 10^{-07}$	3.726	$6.9502 \cdot 10^{-07}$	12.685	$2.7211 \cdot 10^{-07}$	13.080
12	$5.4890 \cdot 10^{-08}$	3.710	$6.8824 \cdot 10^{-08}$	3.744	$1.7972 \cdot 10^{-07}$	12.766	$7.0742 \cdot 10^{-08}$	13.205
13	$1.4716 \cdot 10^{-08}$	3.730	$1.8309 \cdot 10^{-08}$	3.759	$5.3551 \cdot 10^{-08}$	12.979	$2.0194 \cdot 10^{-08}$	13.475
14	$3.9274 \cdot 10^{-09}$	3.747	$4.8530 \cdot 10^{-09}$	3.773	$1.3657 \cdot 10^{-08}$	13.160	$5.2160 \cdot 10^{-09}$	13.563
15	$1.0440 \cdot 10^{-09}$	3.762	$1.2822 \cdot 10^{-09}$	3.785	$3.9338 \cdot 10^{-09}$	13.613	$1.4647 \cdot 10^{-09}$	13.787
16	$2.7652 \cdot 10^{-10}$	3.775	$3.3778 \cdot 10^{-10}$	3.796	$9.9975 \cdot 10^{-10}$	13.660	$3.7657 \cdot 10^{-10}$	13.851
17	$7.3010 \cdot 10^{-11}$	3.787	$8.8755 \cdot 10^{-11}$	3.806	$2.8500 \cdot 10^{-10}$	13.803	$1.0434 \cdot 10^{-10}$	14.038

Figure 3 shows the ratios ($n = \log_2(h_n^{-1})$)

$$\frac{e_{n-1}^c(P_2)/(n-1)}{e_n^c(P_2)/n},$$

i.e. the decrease in $e_n^c(P_2)/n$ proceeding from level $n-1$ to level n for our different model problems. We see the $O(h_n^{1+\alpha}\log_2(h_n^{-1}))$-behaviour for $\alpha \in \{0.5, 0.9\}$ and the $O(h_n^2 \log_2(h_n^{-1}))$-convergence for $\alpha \in \{1.5, 2.5, 3.5\}$ and for our last example with $g(y) = \sin(\pi y)$.

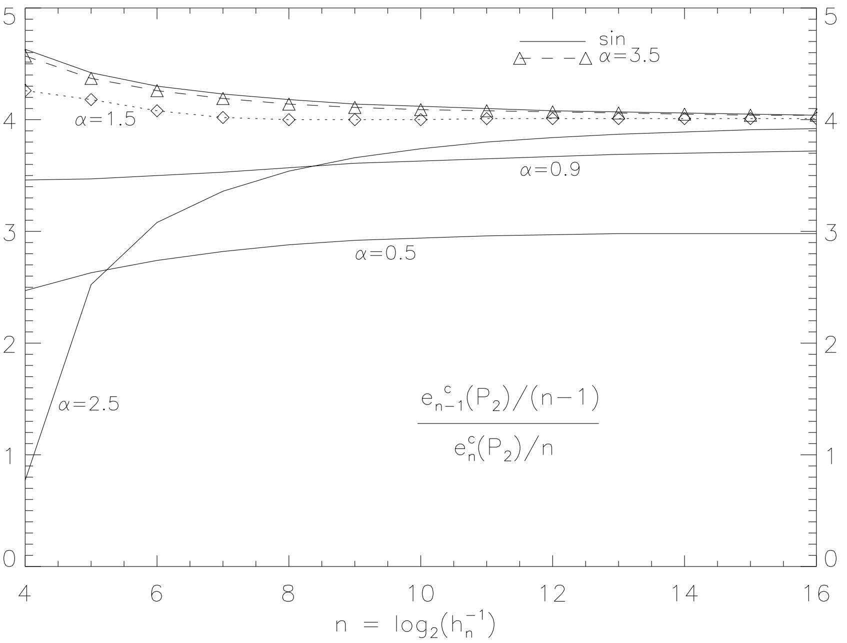

FIG. 3. *Decrease in $e_n^c(P_2)/n$ proceeding from level $n-1$ to level n.*

In figure 4, we see the plots of the pointwise error $e_n^c(P_2)$ divided by h_n^2. For $\alpha \in \{1.5, 2.5, 3.5\}$ and $g(y) = \sin(\pi y)$, we observe a linear dependence on n, which shows again the $O(h_n^2 \log_2(h_n^{-1}))$-behaviour of $e_n^c(P_2)$. However, for $\alpha \in \{0.5, 0.9\}$, the convergence is worse.

8. Concluding Remarks. In this paper, we have dealt with questions of convergence of the combination technique for the Laplacian. We have seen that for sufficiently smooth Dirichlet boundary functions g, i.e., if the Fourier coefficients b_k of the 2-periodic and 0-symmetric extension of g are of the order $O(k^{-3-\varepsilon})$, the pointwise error of the solution resulting from a finite difference discretization on a rectangular grid with meshwidths h_x and h_y allows a splitting of the type

$$u_{h_x,h_y}(x,y) - u(x,y) \;=\; C_1(x,y,h_x)h_x^2 + C_2(x,y,h_y)h_y^2 + C(x,y,h_x,h_y)h_x^2h_y^2$$

with $|C_1(x,y,h_x)|$, $|C_2(x,y,h_y)|$, and $|C(x,y,h_x,h_y)|$ bounded by some positive $B(x,y)$ for all $h_x, h_y \in]0, \frac{1}{2}]$. If $g \in C^4([0,1])$ and $g(0) = g(1) = g''(0) = g''(1) = 0$, the same result recently has been shown by the authors in a different approach based on the interactive use of tools of symbolic computation (see **[2]**). From **[9]**, we know that the above splitting is a sufficient condition for the $O(h_n^2 \log_2(h_n^{-1}))$-behaviour of the error $e_n^c(x,y)$ of the combined solution $u_n^c(x,y)$, which is typical for the sparse grid context.

In several numerical experiments, we have studied the convergence of the pointwise error and of error norms for both smooth and non-smooth boundary functions. Even in the latter situation, the pointwise rates of convergence seem to be only slightly deteriorated by a logarithmic factor that stems from the use of sparse grids.

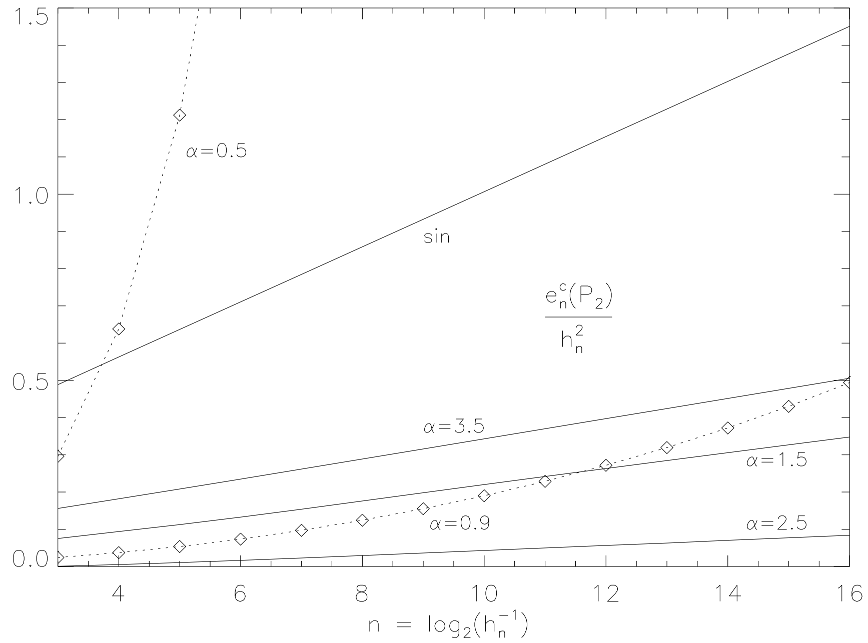

FIG. 4. *Pointwise error* $e_n^c(P_2)$ *divided by* h_n^2.

Furthermore, if we compare our results with those in **[14]** for Richardson extrapolation, we observe the same rates of convergence of the norms. However, the combination technique shows a better behaviour of the pointwise error in the non-smooth case, i.e, if the Dirichlet boundary function does not fulfill the smoothness requirement $b_k < \frac{1}{k^{3+\varepsilon}}$ of theorem 6.

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