# On the abstract theory of additive and multiplicative Schwarz algorithms

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**Abstract.** In recent years, it has been shown that many modern iterative algorithms (multigrid schemes, multilevel preconditioners, domain decomposition methods etc.) for solving problems resulting from the discretization of PDEs can be interpreted as additive (Jacobi-like) or multiplicative (Gauss-Seidel-like) subspace correction methods. The key to their analysis is the study of certain metric properties of the underlying splitting of the discretization space $V$ into a sum of subspaces $V_j$ and the splitting of the variational problem on $V$ into auxiliary problems on these subspaces.

In this paper, we propose a modification of the abstract convergence theory of the additive and multiplicative Schwarz methods, that makes the relation to traditional iteration methods more explicit. The analysis of the additive and multiplicative Schwarz iterations can be carried out in almost the same spirit as in the traditional block-matrix situation, making convergence proofs of multilevel and domain decomposition methods clearer, or, at least, more classical.

In addition, we present a new bound for the convergence rate of the appropriately scaled multiplicative Schwarz method directly in terms of the condition number of the corresponding additive Schwarz operator.

These results may be viewed as an appendix to the recent surveys [X], [Ys].

**Key words:** symmetric positive definite linear systems, iterative methods, additive and multiplicative Schwarz methods, multilevel preconditioning, multigrid methods, domain decomposition

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# 1 Introduction

In recent years, a lot of progress has been made in the design of efficient iterative solvers for large sparse linear systems arising from the discretization of symmetric elliptic boundary value problems. It has turned out that many of these methods for this problem class (multigrid schemes, multilevel preconditioners, domain decomposition algorithms etc.) can be analyzed from a unified viewpoint if they are interpreted as additive or multiplicative Schwarz methods. For an introduction to the existing abstract convergence theory for this type of iterative methods we refer to the excellent survey papers by Xu [X] and Yserentant [Ys] where also some historical information can be found (cf. in addition [W], [DSW], [DW1], [DW2]).

It is the aim of this paper to give a slightly different derivation of the abstract Schwarz theory, and to analyze the assumptions necessary to get sharp condition number estimates for the additive Schwarz operator on the one hand, and bounds for the convergence rate of the multiplicative Schwarz algorithm on the other. Our technical device is an abstract version of the generating (or semidefinite) system used in [G1], [G2] to describe and analyze multilevel Schwarz algorithms. In our opinion, this approach makes the relation to traditional iteration methods more explicit. The analysis of the additive and multiplicative Schwarz iterations can be carried out in almost the same spirit as in the traditional block-matrix situation, making convergence proofs of multilevel and domain decomposition methods clearer, or, at least, more classical. Furthermore, we present a new bound for the convergence rate of the appropriately scaled multiplicative Schwarz method directly in terms of the condition number of the corresponding additive Schwarz operator.

The remainder of this paper is organized as follows: In section 2 we introduce three formulations for our discretized symmetric PDE to be solved, the conventional weak formulation, the additive Schwarz formulation that is based on some decomposition of the underlying Hilbert space into subspaces [DSW], [DW3], [X], [Ys], [Z] and a new, "enlarged" Schwarz formulation by means of the cartesian product of Hilbert spaces arising in the particular space decomposition. We discuss the relations between these three formulations.

In Section 3 we study the additive and multiplicative Schwarz iteration associated with a given subspace decomposition. It turns out that, with the help of the "enlarged" formulation, these methods can be rewritten as classical Jacobi-/Richardson- and Gauss-Seidel-/SOR-like iterations, respectively. Also, the conditions for convergence and the terms entering the convergence estimates are analogous to those of classical iteration methods. Even the convergence proofs themselves follow basically the same lines as the respective proofs for the classical iteration methods.

In Section 4 we present a new bound for the convergence rate of the appropriately scaled multiplicative Schwarz method directly in terms of the condition number of the corresponding additive Schwarz operator.

## 2 Space decomposition and problem reformulations

We use the following Hilbert space setting; the matrix and operator notations used in [X], [Ys], respectively, can easily be derived from it. Let $V$ be some fixed, finite-dimensional Hilbert space. The scalar product in $V$ is denoted by $(\cdot,\cdot)$. We consider a positive definite, symmetric bilinear form $a(u,v)=(Au,v)$, $u,v\in V$, with $A:V\to V$ denoting the corresponding s.p.d. operator acting on $V$.

Now, consider an arbitrary additive representation of $V$ by the sum of a finite number of subspaces $V_j\subset V$ :

$$V=\sum_{j=0}^{J} V_j \ . \tag{1}$$

More precisely, this means that any $u\in V$ possesses at least one representation $u=\sum_{j=0}^{J} u_j$ where $u_j\in V_j$ for all $j=0,\ldots,J$. Suppose that the $V_j$ are equipped with auxiliary continuous s.p.d. forms $b_j(u_j,v_j)=(B_ju_j,v_j)$ given by the s.p.d. operators $B_j:V_j\to V_j$. These forms might model approximative solvers used on the subspaces, i.e. $B_j^{-1}$ is an approximative inverse for $A_j$ the restriction of $A$ to $V_j$.

From now on, it will be convenient to use $(\cdot,\cdot)_V=a(\cdot,\cdot)$ as the basic scalar product of $V$ which we will sometimes indicate by writing $\{V;a\}$ instead of just $V$. The same holds true for $\{V_j;b_j\}$. For the splitting

$$\{V;a\}=\sum_{j=0}^{J}\{V_j;b_j\}\ , \tag{2}$$

we define a norm $|||.|||$ on $V$ by

$$|||u|||^2=\inf_{u_j\in V_j\,:\,u=\sum_{j=0}^{J}u_j}\ \sum_{j=0}^{J} b_j(u_j,u_j)\ , \tag{3}$$

and introduce the positive and finite values

$$\lambda_{\min}=\inf_{u\in V,u\neq 0}\frac{a(u,u)}{|||u|||^2}\ ,\quad \lambda_{\max}=\sup_{u\in V,u\neq 0}\frac{a(u,u)}{|||u|||^2}\ . \tag{4}$$

The quantity

$$\kappa\equiv\kappa(\{V;a\};\sum_{j=0}^{J}\{V_j,b_j\})=\frac{\lambda_{\max}}{\lambda_{\min}} \tag{5}$$

will be called condition number of the splitting (2). Note that the condition number does not change if we change the order of the subspaces. For the case of an infinite-dimensional $V$, this approach can be generalized to the concept of so-called stable splittings, compare [O3].

We now introduce the Hilbert space

$$\tilde{V} = \{\tilde{u} = \{u_j\} \,:\, \sum_{j=0}^{J} b_j(u_j,u_j) < \infty\} \;, \quad \tilde{b}(\tilde{u},\tilde{v}) \equiv (\tilde{u},\tilde{v})_{\tilde{V}} = \sum_{j=0}^{J} b_j(u_j,v_j) \qquad (6)$$

which is nothing but the cartesian product of the Hilbert spaces $V_j$, i.e. $\tilde{V} = V_0 \times \ldots \times V_J$ where each subspace $V_j$ is equipped with the auxiliary s.p.d. form $b_j(.,.)$. Furthermore, to link $\tilde{V}$ to $V$, we consider the operator

$$R \,:\, \tilde{u} \in \tilde{V} \longmapsto u \equiv R\tilde{u} = \sum_{j=0}^{J} u_j \;. \qquad (7)$$

Obviously, for our splitting into a finite number of subspaces, $R$ is a linear bounded operator from $\tilde{V}$ onto $V$. Moreover, since

$$a(R\tilde{u},R\tilde{u}) = a(\sum_{j=0}^{J} u_j, \sum_{j=0}^{J} u_j) \leq \lambda_{\max} \cdot \sum_{j=0}^{J} b_j(u_j,u_j) = \lambda_{\max} \cdot (\tilde{u},\tilde{u})_{\tilde{V}}$$

we see that

$$\|R\|_{\tilde{V}\to V} \leq \sqrt{\lambda_{\max}} \;. \qquad (8)$$

We now introduce the adjoint operator by

$$R^* \,:\, u \in V \longmapsto \tilde{u} \equiv R^* u = \{T_j u\} \in \tilde{V} \;, \qquad (9)$$

with the operators $T_j \,:\, V \to V_j$ given by the variational problems

$$b_j(T_j u, v_j) = a(u, v_j) \qquad \forall\, v_j \in V_j \;. \qquad (10)$$

In operator notation $T_j = B_j^{-1} Q_j A$ where $Q_j : V \to V_j$ denotes the orthoprojection onto $V_j$ with respect to the scalar product $(\cdot,\cdot)$. Indeed, the definition of the adjoint operator is satisfied, i.e. we have

$$(R^* u, \tilde{u})_{\tilde{V}} = \sum_{j=0}^{J} b_j(T_j u, u_j) = \sum_{j=0}^{J} a(u,u_j) = a(u, \sum_{j=0}^{J} u_j) = (u, R\tilde{u})_V \;.$$

Finally, to any linear continuous functional $\Phi$ on $V$, we associate the elements $\tilde{\phi} \in \tilde{V}$ and $\phi = R\tilde{\phi} \in V$ by defining the $\phi_j \in V_j$ via the variational problems

$$b_j(\phi_j, v_j) = \Phi(v_j) \qquad \forall\, v_j \in V_j \;. \qquad (11)$$

which are analogous to (10).

With these preparations completed, we come to the main result of this section. We look for the solution of a variational problem associated with $a(\cdot,\cdot)$.

The standard weak formulation is:

*Find* $u \in V$ *such that*
$$a(u,v) = \Phi(v) \qquad \forall\, v \in V\ . \tag{12}$$
We can rewrite this problem as
*Find* $u \in V$ *such that*
$$Pu = \phi \qquad (\ P \equiv RR^* = \sum_{j=0}^{J} T_j\ :\ V \to V\ )\ . \tag{13}$$
Alternatively, we can also reformulate our problem as
*Find* $u = R\tilde{u} \in V$ *where* $\tilde{u} \in \tilde{V}$ *satisfies*
$$\tilde{P}\tilde{u} = \tilde{\phi} \qquad (\ \tilde{P} \equiv R^*R\ :\tilde{V} \to \tilde{V}\ )\ . \tag{14}$$

The following simple Theorem 1 states the equivalence of all three formulations and gives some important properties of the operators $P$ and $\tilde{P}$.

**Theorem 1**. *Suppose that $V$ is finite-dimensional, and that the splitting* (2) *is finite. Let the characteristic numbers $\lambda_{\max}, \lambda_{\min}$, and $\kappa$ of the splitting be given by* (4), (5).

(a) *The problems* (12), (13), *and* (14) *have the same (unique) solution.*

(b) *The operator $P = RR^* = \sum_{j=0}^{J} T_j$ is symmetric positive definite on $V$ w.r.t. $(\cdot,\cdot)_V$ and the minimal interval containing its spectrum is given by the constants from* (4):
$$\mathrm{spectrum}(P) \subset [\inf_{u\in V\,:\,a(u,u)=1} a(Pu,u), \sup_{u\in V\,:\,a(u,u)=1} a(Pu,u)] = [\lambda_{\min}, \lambda_{\max}] \tag{15}$$
*Thus, the condition number $\kappa(P) = \|P\|_{V\to V}\cdot\|P^{-1}\|_{V\to V}$ of $P$ coincides with the condition number $\kappa$ of the splitting* (2) *defined by* (5).

(c) *Analogously, the operator $\tilde{P} = R^*R$ is symmetric positive semi-definite on $\tilde{V}$ w.r.t. $(\cdot,\cdot)_{\tilde{V}}$. Its null-space* $\mathrm{Ker}(P) = \mathrm{Ker}(R)$ *is trivial iff the representation* (1) *is a direct sum. The spectrum of $\tilde{P}$ essentially coincides with that of $P$:*
$$\mathrm{spectrum}(\tilde{P}) = \{0\} \cup \mathrm{spectrum}(P). \tag{16}$$

This theorem is simple but useful : 1(b) provides conditions on the splitting (2) and the choice for the forms $b_j$ under which the new problem (13) is well-conditioned. The idea is then to solve (13) by a Richardson- or Conjugate-Gradient-type iteration which should lead to fast convergence. Since
$$P = RR^* = \sum_{j=0}^{J} T_j = \left(\sum_{j=0}^{J} B_j^{-1} Q_j\right) A \equiv C \cdot A\ , \tag{17}$$

the switching to formulation (13) can be viewed as preconditioning strategy with the preconditioner $C$ for the original problem (12).

In the same way, Theorem 1(c) gives us conditions on the splitting (2) and the choice for the forms $b_j$ such that the new enlarged problem (14) is well-conditioned. Now, again, the idea is to find just one of the many non-unique solutions $\tilde{u}$ of (14) by a Richardson- or Conjugate-Gradient-type iteration applied directly to $\tilde{P}\tilde{u} = \tilde{\phi}$ which should lead to fast convergence. Additionally any other iterative method, such as Gauss-Seidel- or a SOR-like iteration can be used directly on $\tilde{P}\tilde{u} = \tilde{\phi}$ as well.

The proof of Theorem 1(a) is obvious; we leave it to the reader. Theorem 1(b) has many authors, compare for example [MN], [BM], [W], [Z]. Its proof can be given in a few lines using the explicit formula

$$a(P^{-1}u, u) = |||u|||^2 \quad , \qquad u \in V,$$

(see, e.g. [W], [X], [GO1]). Theorem 1(c) was first shown in a slightly more algebraic setting in [G1]. In our notation, we have

$$(\tilde{P}\tilde{u}, \tilde{v})_{\tilde{V}} = (R^* R\tilde{u}, \tilde{v})_{\tilde{V}} = (R\tilde{u}, R\tilde{v})_V = (R\tilde{v}, R\tilde{u})_V = (R^* R\tilde{v}, \tilde{u})_{\tilde{V}} = (\tilde{u}, \tilde{P}\tilde{v})_{\tilde{V}}$$

which shows that $\tilde{P}$ is symmetric on $\tilde{V}$. Furthermore, the relation

$$\tilde{b}(\tilde{P}\tilde{u}, \tilde{u}) \equiv (\tilde{P}\tilde{u}, \tilde{u})_{\tilde{V}} = (R^* R\tilde{u}, \tilde{u})_{\tilde{V}} = (R\tilde{u}, R\tilde{u})_V \geq 0$$

shows that $\tilde{P}$ is positive semi-definite, and makes clear that $\mathrm{Ker}(\tilde{P}) = \mathrm{Ker}(R)$. Now, let $\lambda \in \mathrm{spectrum}(P)$. Then $\exists u \in V, u \neq 0$ with $RR^* u = \lambda u$, i.e. $R^* R(R^* u) = \lambda(R^* u)$. But $R^* u \neq 0$ since $\mathrm{Ker}(R^*) = \{0\}$. Thus, $\lambda \in \mathrm{spectrum}(\tilde{P})$. Conversely, if $\lambda \in \mathrm{spectrum}(\tilde{P})$ then $R^* R\tilde{u} = \lambda\tilde{u}$, $\tilde{u} \in \tilde{V}, \tilde{u} \neq \tilde{0}$ which gives $RR^*(R\tilde{u}) = \lambda(R\tilde{u})$. If $\lambda \neq 0$ then $R\tilde{u} \neq 0$, i.e. $\lambda \in \mathrm{spectrum}(P)$. The case $\lambda = 0$ is possible iff the splitting is overlapping. Then, its eigenspace is $\mathrm{Ker}(R)$. $\square$

Theorem 1(b) and 1(c) can also be obtained from the fictitious space lemma (see [N1], [N2]) which has been known in the Russian literature for some years. This lemma also shows the close relation between formulation (13) and (14).

**Fictitious Space Lemma.** *Let $H_0$ and $H$ be two Hilbert spaces, with the scalar products denoted by $(\cdot,\cdot)_0$ and $(\cdot,\cdot)$, respectively, and with bilinear forms $a : H_0 \times H_0 \to$ $\mathbf{R}$ and $b : H \times H \to \mathbf{R}$, respectively, generated by the s.p.d. operators $A : H_0 \to H_0$ and $B : H \to H$, respectively. Suppose that there is a surjective linear operator $R :$ $H \to H_0$, such that for all $u_0 \in H_0$ there exists a $v \in H$ with $Rv = u_0$ and*

$$c_T \cdot b(v, v) \leq a(u_0, u_0) \ ,$$

*and also*

$$a(Ru, Ru) \leq c_R \cdot b(u, u) \qquad \forall\, u \in H \ .$$

*Introduce the adjoint operator $R^* : H_0 \to H$ by $(Ru, u_0)_0 = (u, R^* u_0)$ for all $u \in H$ and $u_0 \in H_0$. Then*

$$c_T \cdot a(u_0, u_0) \le a(RB^{-1}R^*Au_0, u_0) \le c_R \cdot a(u_0, u_0) , \quad \forall\, u_0 \in H_0 ,$$

*with sharp bounds for the spectrum of $RB^{-1}R^*A$ given by the best possible constants $c_T$ and $c_R$ in the above inequalities.*

Proof of the fictitious space lemma (see [N2]): For any $u_0 \in H_0$, we have

$$\begin{aligned} a(RB^{-1}R^*Au_0, u_0) &\le a(RB^{-1}R^*Au_0, RB^{-1}R^*Au_0)^{1/2} \cdot a(u_0, u_0)^{1/2} \\ &\le c_R^{1/2} \cdot b(B^{-1}R^*Au_0, B^{-1}R^*Au_0)^{1/2} \cdot a(u_0, u_0)^{1/2} \\ &= c_R^{1/2} \cdot a(RB^{-1}R^*Au_0, u_0)^{1/2} \cdot a(u_0, u_0)^{1/2} , \end{aligned}$$

which gives the upper bound. We also have

$$\begin{aligned} a(u_0, u_0) &= (Rv, Au_0)_0 = b(v, B^{-1}R^*Au_0) \\ &\le b(v, v)^{1/2} \cdot b(B^{-1}R^*Au_0, B^{-1}R^*Au_0)^{1/2} \\ &\le c_T^{-1/2} \cdot a(u_0, u_0)^{1/2} \cdot a(RB^{-1}R^*Au_0, u_0)^{1/2} , \end{aligned}$$

which yields the lower bound. The sharpness of the bounds is obvious. □

Alternatively, if we look carefully at the conditions and the proof, we easily obtain the identity

$$a((RB^{-1}R^*A)^{-1}u_0, u_0) = \inf_{v \in H\,:\, Rv = u_0} b(v, v) , \quad \forall\, u_0 \in H_0 , \tag{18}$$

which also implies the above assertions. Indeed, let $X = (RB^{-1}R^*A)^{-1}$ and let $v$ be an arbitrary element of $H$ such that $Rv = u_0$. We get

$$a(Xu_0, u_0) = (AXu_0, Rv) = b(\underbrace{B^{-1}R^*AXu_0}_{w}, v) \le b(w, w)^{1/2} \cdot b(v, v)^{1/2}$$

with equality for $v = w$. (Check that $Rw = u_0$ by definition of $X$). To obtain (18), it remains to take the infimum with respect to all admissible $v$, and to observe that

$$b(w, w) = (B^{-1}R^*AXu_0, R^*AXu_0) = (X^{-1}Xu_0, AXu_0)_0 = a(Xu_0, u_0).$$

As in Theorem 1, one could write down explicit formulae for the minimal and maximal eigenvalues of $RB^{-1}R^*A$.

Now, to see that Theorem 1 is a consequence of this lemma, choose $H_0 = V$ with the originally given scalar product $(\cdot, \cdot)$, and let

$$H = \tilde{V} = V_0 \times \cdots \times V_J , \quad (\tilde{u}, \tilde{v})_0 = \sum_{j=0}^{J} (u_j, v_j) .$$

As above, $\tilde{u} \equiv (u_0, \ldots, u_J)$ denotes an arbitrary element of $V_0 \times \cdots \times V_J$. The bilinear forms on $H_0 = V$ and $H = \tilde{V}$, respectively, are $a(\cdot,\cdot)$ and $b(\cdot,\cdot) = \tilde{b}(\cdot,\cdot)$, respectively, while $R$ has the same meaning as before.

With these choices, $RB^{-1}R^*A$ actually coincides with the operator $P$ in (17) and (18) leads to the assertion of Theorem 1(b).

Note that the formulation (13) is nothing but the additive Schwarz formulation of (12), which has already been treated by many authors, see [X], [Ys], [DSW], [Z] for further references.

The reformulation of (12) given by (14) is essentially the abstract version of the semidefinite linear system introduced in [G1], [G2] in terms of a generating system. In the next section, it will be useful in connection with the treatment of the additive and multiplicative iterative methods as Jacobi- and Gauss-Seidel-like methods, respectively.

# 3 Additive and multiplicative Schwarz methods

Now we turn to the additive and multiplicative variants of the Schwarz iteration associated with the splitting (2).

The additive subspace correction algorithm [X] associated with the splitting (2) is defined as the Richardson method applied to (13):

$$u^{(l+1)} = u^{(l)} - \omega \cdot (Pu^{(l)} - \phi) = u^{(l)} - \omega \cdot \sum_{j=0}^{J} (T_j u^{(l)} - \phi_j) \,, \quad l = 0, 1, \ldots \,. \tag{19}$$

Here $u^{(0)} \in V$ is any given initial approximation to the solution $u$ of (12) and (13), respectively, and $\omega$ is a relaxation parameter. In contrast to the parallel incorporation of the subspace corrections $r_j^{(l)} = T_j u^{(l)} - \phi_j$ into the iteration (19), the multiplicative algorithm uses them in a sequential way:

$$v^{(l+(j+1)/(J+1))} = v^{(l+j/(J+1))} - \omega \cdot (T_j v^{(l+j/(J+1))} - \phi_j) \,, \quad j = 0, \ldots, J \,,\; l = 0, 1, \ldots \,. \tag{20}$$

It is worth interpreting the iterations (19) and (20) using the formulation (14). To this end, we use the matrix representation of $\tilde{P}$ with respect to the 'coordinate' spaces $V_j$ of $\tilde{V} = V_0 \times V_1 \times \ldots \times V_J$ and decompose it into lower triangular, diagonal, and upper triangular parts:

$$\tilde{P} = \{T_{i,j}\}_{i,j=0}^{J} = \tilde{L} + \tilde{D} + \tilde{U} \,, \quad T_{i,j} \equiv T_i|_{V_j} : V_j \to V_i \tag{21}$$

where $L_{j,i} = T_{j,i}$ for $j < i$, $D_{i,i} = T_{i,i}$ and $U_{i,j} = T_{i,j}$ for $j > i$ while all remaining entries are zero operators between the respective subspaces. Note also that $\tilde{U} = \tilde{L}^*$ since

$$b_i(T_{i,j}u_j, v_i) = b_i(T_i u_j, v_i) = a(u_j, v_i) = \ldots = b_j(u_j, T_{j,i} v_j) \,.$$

If we define the Richardson (or Jacobi-like) iteration for (14) by

$$\tilde{u}^{(l+1)} = \tilde{u}^{(l)} - \omega \cdot (\tilde{P}\tilde{u}^{(l)} - \tilde{\phi}) \,, \quad l = 0, 1, \ldots \tag{22}$$

and the Gauss-Seidel-like (or better SOR-like) iteration by

$$(\tilde{I} + \omega \cdot \tilde{L})\tilde{v}^{(l+1)} = (\tilde{I} - \omega \cdot (\tilde{D} + \tilde{U}))\tilde{v}^{(l)} + \omega \cdot \tilde{\phi} \,, \quad l = 0, 1, \ldots \tag{23}$$

then

$$u^{(l)} = R\tilde{u}^{(l)} \,, \qquad v^{(l)} = R\tilde{v}^{(l)} \,, \; l = 1, 2, \ldots \tag{24}$$

whenever this relation is satisfied for the starting approximations, i.e. for $l = 0$. Here, $\tilde{I}$ denotes the identity operator on $\tilde{V}$. Moreover, for any iteration method in $\tilde{V}$ described by

$$\tilde{u}^{(l+1)} = \tilde{u}^{(l)} - \tilde{N}\tilde{P}\tilde{u}^{(l)} + \tilde{N}\tilde{\phi} \tag{25}$$

with some given operator $\tilde{N}$ in $\tilde{V}$, we obtain an iterative method in $V$ given by

$$u^{(l+1)} = u^{(l)} - R\tilde{N}R^{*}u^{(l)} + R\tilde{N}\tilde{\phi} \tag{26}$$

with the relation $u^{(l+1)} = R\tilde{u}^{(l+1)}$ fulfilled, whenever it holds for $l = 0$.

In our example, we have to take

$$\tilde{N} = \omega\tilde{I} \tag{27}$$

for the additive scheme, and

$$\tilde{N} = (\frac{1}{\omega}\tilde{I} + \tilde{L})^{-1} \tag{28}$$

for the multiplicative scheme (also c.f. (33)). While the verification is trivial for the iteration (19) and (22), respectively, it takes a little bit algebra to check the identity

$$I - R(\frac{1}{\omega}\tilde{I} + \tilde{L})^{-1}R^{*} = (I - \omega T_J) \ldots (I - \omega T_0)$$

for the multiplicative case (20) and (23), respectively. We leave this as an exercise to the reader.

A consequence of this observation, which was made in [G1], [G2], [GO2], is that the analysis of the methods (19), (20) can be carried out in almost the same spirit as in the traditional block-matrix situation if one uses the formulation (14). The sometimes tricky proofs, including the original one in [BPWX1], [BPWX2], [BP] and [X], (see also the comments on this point in [Ys]) for the multiplicative case can be made clearer, or, at least, more classical.

The following results, which also explain the central role of the above splitting concept, are derived from [GO2]; see [H] for statements of this type in the matrix case.

**Theorem 2 (Additive Schwarz iteration).** *Suppose that $V$ is finite-dimensional, and that the splitting* (2) *is finite. Let the characteristic numbers $\lambda_{\max}, \lambda_{\min}$, and $\kappa$ of the splitting be given by* (4), (5).

(a) *The additive method* (19) *converges for $0 < \omega < 2/\lambda_{\max}$, with the convergence rate*

$$\rho_{as} = \max\{|1-\omega\cdot\lambda_{\min}|, |1-\omega\cdot\lambda_{\max}|\}\ . \tag{29}$$

(b) *The bound in (29) takes the minimum*

$$\rho_{as}^* = 1 - \frac{2}{1+\kappa} \quad \textit{for} \quad \omega^* = \frac{2}{\lambda_{\max}+\lambda_{\min}}. \tag{30}$$

Proof of Theorem 2: The error propagation operator of (19) (in $V$) is $M_{as} \equiv I - \omega\cdot P$ where $I$ denotes the identity in $V$. Now, for $0 < \omega < 2/\lambda_{\max}$ we have $-1 < 1-\omega\lambda_{\max} \leq 1-\omega\lambda_{\min} < 1$ and with (29) we obtain $\rho(M_{as}) < 1$, i.e. convergence.

If, on the other hand, we assume convergence, i.e. $\rho(M_{as}) < 1$, we get from $1 > \lambda_{\max}(M_{as}) \geq |1-\omega\lambda_{\max}| \geq 1-\omega\lambda_{\max}$ that $\omega\lambda_{\max} > 0$ and consequently $\omega > 0$ since $\lambda_{\max} > 0$. ($P$ has a positive real spectrum). Analogously, the relation $-1 < -\rho(M_{as}) \leq -|1-\omega\lambda_{\max}| \leq 1-\omega\lambda_{\max}$ leads to $\omega\lambda_{\max} < 2$, which gives directly $\omega < 2/\lambda_{\max}$.

Now, the optimal value $\omega^*$ is just the intersection of the lines $y(\omega) = \omega\lambda_{\max} - 1$ and $z(\omega) = 1-\omega\lambda_{\min}$. This leads to (30). □

Note that this proof is exactly the same as for the traditional Richardson-iteration, compare e.g. [H]. Note also that, even in the case of divergence, the additive Schwarz iteration serves as an optimal preconditioner as long as $\kappa$ is independent of $J$.

**Theorem 3 (Multiplicative Schwarz iteration).** *Suppose that $V$ is finite-dimensional, and that the splitting* (2) *is finite. Let the characteristic numbers $\lambda_{\max}, \lambda_{\min}$, and $\kappa$ of the splitting be given by* (4), (5)*. Furthermore, let $\|\tilde{L}\|_{\tilde{V}\to\tilde{V}}$ be the norm of the lower triangular matrix operator $\tilde{L}$ (cf.* (21)*) as an operator in $\tilde{V}$, let*

$$\omega_1 \equiv \lambda_{\max}(\tilde{D}) = \max_{j=0,\ldots,J}\left\{\max_{u_j\in V_j}\frac{a(u_j,u_j)}{b_j(u_j,u_j)}\right\}$$

*and let $\tilde{W} := \frac{1}{\omega}\tilde{I} + \tilde{L}$.*

(a) *The multiplicative method* (20) *converges for $0 < \omega < 2/\omega_1$, with a bound for the asymptotic convergence rate given by*

$$\rho_{ms} \leq \sqrt{1 - \frac{\lambda_{\min}\cdot(\frac{2}{\omega}-\omega_1)}{\|\tilde{W}\|^2_{\tilde{V}\to\tilde{V}}}} \leq \sqrt{1 - \frac{\lambda_{\min}\cdot(\frac{2}{\omega}-\omega_1)}{(\frac{1}{\omega}+\|\tilde{L}\|_{\tilde{V}\to\tilde{V}})^2}}\ . \tag{31}$$

(b) *The bound in* (31) *takes its minimum*

$$\rho_{ms}^* \leq \sqrt{1 - \frac{\lambda_{\min}}{2\|\tilde{L}\|_{\tilde{V}\to\tilde{V}} + \omega_1}} \qquad \textit{for} \qquad \omega^* = \frac{1}{\|\tilde{L}\|_{\tilde{V}\to\tilde{V}} + \omega_1}. \tag{32}$$

Now, we see a difference between the additive and multiplicative method. For the additive method, basically the terms $\lambda_{\min}$ and $\lambda_{\max}$ enter the convergence rate whereas for the multiplicative method the terms $\lambda_{\min}$ and $||\tilde{L}||_{\tilde{V}\to\tilde{V}}$ are involved. Note that $||\tilde{L}||_{\tilde{V}\to\tilde{V}}$ depends on the ordering of the successive subspace corrections.

Proof of Theorem 3(a): Since

$$\begin{aligned} R\tilde{v}^{(l+1)} &= R(\tilde{I} + \omega\tilde{L})^{-1}(\tilde{I} - \omega(\tilde{D} + \tilde{U})v^{(l)} + R(\tilde{I} + \omega\tilde{L})^{-1}\tilde{\phi} \\ &= R(\tilde{I} - \omega(\tilde{I} + \omega\tilde{L})^{-1}R^*R)v^l + R(\tilde{I} + \omega\tilde{L})^{-1}\tilde{\phi} \\ &= (I - R(\frac{1}{\omega}\tilde{I} + \tilde{L})^{-1}R^*)Rv^{(l)} + R(\tilde{I} + \omega\tilde{L})^{-1}\tilde{\phi} \end{aligned}$$

we obtain the formula

$$M_{ms} = I - R(\frac{1}{\omega}\tilde{I} + \tilde{L})^{-1}R^* \tag{33}$$

for the error propagation operator of the multiplicative Schwarz method in $V$; c.f. also (28).

We have

$$\begin{aligned} M_{ms}^T M_{ms} &= (I - R(\frac{1}{\omega}\tilde{I} + \tilde{L}^T)^{-1}R^*)(I - R(\frac{1}{\omega}\tilde{I} + \tilde{L})^{-1}R^* \\ &= I - R(\frac{1}{\omega}\tilde{I} + \tilde{L}^T)^{-1}\left\{\frac{2}{\omega}\tilde{I} + \tilde{L} + \tilde{L}^T - R^*R\right\}(\frac{1}{\omega}\tilde{I} + \tilde{L})^{-1}R^* \\ &= I - R\tilde{W}^{-T}(\frac{2}{\omega}\tilde{I} - \tilde{D})\tilde{W}^{-1}R^* \, . \end{aligned}$$

Thus,

$$\frac{a(M_{ms}u, M_{ms}u)}{a(u,u)} = 1 - \frac{a(R\tilde{W}^{-T}(\frac{2}{\omega}\tilde{I} - \tilde{D})\tilde{W}^{-1}R^*u, u)}{a(u,u)} \, . \tag{34}$$

But, by the definition of $\lambda_{\min}$ and the property $P = RR^*$ it follows that

$$a(u,u) \leq \frac{1}{\lambda_{\min}}a(Pu,u) = \frac{1}{\lambda_{\min}}a(RR^*u,u) = \frac{1}{\lambda_{\min}}\tilde{b}(R^*u, R^*u)$$

and

$$\begin{aligned} a(R\tilde{W}^{-T}(\frac{2}{\omega}\tilde{I} - \tilde{D})\tilde{W}^{-1}R^*u, u) &= \tilde{b}((\frac{2}{\omega}\tilde{I} - \tilde{D})\tilde{W}^{-1}R^*u, \tilde{W}^{-1}R^*u) \\ &\geq \lambda_{\min}(\frac{2}{\omega}\tilde{I} - \tilde{D}) \cdot ||\tilde{W}||_{\tilde{V}\to\tilde{V}}^{-2} \cdot \tilde{b}(R^*u, R^*u) \\ &\geq \frac{(\frac{2}{\omega} - \omega_1)\lambda_{\min}}{\|\tilde{W}\|_{\tilde{V}\to\tilde{V}}^2}\, a(u,u) \end{aligned}$$

whenever

$$\lambda_{\min}(\frac{2}{\omega}\tilde{I} - \tilde{D}) = \frac{2}{\omega} - \lambda_{\max}(\tilde{D}) = \frac{2}{\omega} - \omega_1 > 0 \ . \tag{35}$$

Substituting into (34), we obtain the relation (31) (left) together with the convergence condition (35). With the estimate

$$||\tilde{W}||^2_{\tilde{V}\to\tilde{V}} = ||\frac{1}{\omega}\tilde{I} + \tilde{L}||^2_{\tilde{V}\to\tilde{V}} \leq (\frac{1}{\omega} + ||\tilde{L}||_{\tilde{V}\to\tilde{V}})^2 \ ,$$

we finally obtain the right inequality of (31). □

Note that the right inequality of (31) basically gives the same estimate as derived in [GO2]. There, the proof relied on Xu's Fundamental Theorem II [X]. Now, however, we have been able to give a straightforward proof that follows the same ideas as the proof for the convergence estimate of a conventional SOR method, see e.g [H], pp. 96-101, compare also [Yo], p. 464. There, a certain matrix $W$ and a splitting of $M^T M$ into the identity and a term of the form $W^{-T} D W^{-1}$ are used, where $D$ is a diagonal matrix. We believe that this will also hold in the same way for other traditional iteration methods with more general iteration matrices $\tilde{N}$; c.f. (25).

Proof of Theorem 3(b): We want to minimize the right estimate in (31). Thus, with $x := 1/\omega$, we have to determine $x$ that maximizes

$$F(x) = \frac{2x - \omega_1}{(x + \|\tilde{L}\|_{\tilde{V}\to\tilde{V}})^2}$$

under the constraint $0 < 1/x < 2/\omega_1$, i.e. $\omega_1/2 < x$. The necessary condition $F'(x) = 0$ gives $x^* = \|\tilde{L}\|_{\tilde{V}\to\tilde{V}} + \omega_1$ which satisfies the constraint, since $\|\tilde{L}\|_{\tilde{V}\to\tilde{V}}$ and $\omega_1$ are positive. A short calculation shows that $F''(x^*) < 0$ which gives the sufficient condition for $x^*$ to be a maximum. With $\omega^* = 1/x^*$, we obtain the desired result. □

The quantity $\|L\|_{\tilde{V}\to\tilde{V}}$ can be further estimated under additional assumptions, e.g. if strengthened Cauchy-Schwarz inequalities are valid for the splitting (see [X], [Ys], [DSW]). If, as in [X], one assumes the Cauchy-Schwarz inequality in the form

$$a(u_k, v_l) \leq \gamma_{kl}\sqrt{b_k(u_k, u_k)}\sqrt{b_l(v_l, v_l)} \quad \forall u_k \in V_k, \forall v_l \in V_l$$

and denotes by $K_2$ the spectral radius of the symmetric, non-negative matrix $\|\gamma_{kl}\|_{k,l=0}^J$, then

$$\begin{aligned} \|\tilde{L}\|_{\tilde{V}\to\tilde{V}} &= \sup_{\|\tilde{u}\|_{\tilde{V}}\leq 1, \|\tilde{v}\|_{\tilde{V}}\leq 1} |(\tilde{L}\tilde{u}, \tilde{v})_{\tilde{V}}| \\ &= \sup_{\|\tilde{u}\|_{\tilde{V}}\leq 1, \|\tilde{v}\|_{\tilde{V}}\leq 1} |\sum_{j=1}^{J}\sum_{l=0}^{j-1} b_j(T_j u_l, v_j)| \end{aligned}$$

$$
\begin{aligned}
&= \sup_{\|\tilde u\|_{\tilde V}\le 1, \|\tilde v\|_{\tilde V}\le 1} |\sum_{j=1}^{J}\sum_{l=0}^{j-1} a(u_l, v_j)| \\
&\le \sup_{\|\tilde u\|_{\tilde V}\le 1, \|\tilde v\|_{\tilde V}\le 1} |\sum_{j=1}^{J}\sum_{l=0}^{J} \gamma_{lj}\sqrt{b_l(u_l,u_l)}\sqrt{b_j(v_j,v_j)}| \\
&\le \sup_{\|\tilde u\|_{\tilde V}\le 1, \|\tilde v\|_{\tilde V}\le 1} K_2 \|\tilde u\|_{\tilde V}\|\tilde v\|_{\tilde V} = K_2.
\end{aligned}
$$

For certain splittings (2), $K_2$ can be estimated from above by a constant independent of $J$. Note that $K_2$ is an upper bound for *all* possible $\|\tilde L\|_{\tilde V\to\tilde V}$ that arise for all possible traversal orderings of the multiplicative scheme. Furthermore, $K_2$ is also an upper bound for the maximal eigenvalue $\lambda_{\max}$ of $\tilde P$. Now, the convergence rate of the multiplicative method as well as the additive method is independent of $J$, if, in addition, the splitting (2) has the property that $\lambda_{\min}$ can be estimated from below by a positive constant that is independent of $J$. However, without any additional assumptions, we obtain the estimate of the following section, that links the terms of the multiplicative method to that of the additive method.

# 4 On the relation between the additive and the multiplicative method

In this section, we give an estimate of $\|\tilde L\|_{\tilde V\to\tilde V}$ in terms of $\lambda_{\max}$ without any additional assumption. This links the estimates of the convergence rate of the multiplicative method to that of the additive method.

**Theorem 4.** *Suppose that $V$ is finite-dimensional, and that the splitting* (2) *is finite. Let the characteristic numbers $\lambda_{\max}$ and $\lambda_{\min}$ of the splitting be given by* (4)*. Let $\tilde P = \tilde L + \tilde D + \tilde L^*$ be the (extended) additive Schwarz operator corresponding to the splitting* (2)*. Then*

$$||\tilde L||_{\tilde V\to\tilde V} \le \frac{1}{2}[\log_2(2J)]\cdot\lambda_{\max}\,. \tag{36}$$

*Thus, without any additional assumptions, we obtain the bound*

$$\rho_{ms} \le \sqrt{1 - \frac{\lambda_{\min}\cdot(\frac{2}{\omega}-\omega_1)}{(\frac{1}{\omega}+\frac{1}{2}[\log_2(2J)]\cdot\lambda_{\max})^2}} \tag{37}$$

*and, with a proper choice of $\omega$, the bound*

$$\rho^*_{ms} \le \sqrt{1-\frac{1}{\log_2(4J)\cdot\kappa}}. \tag{38}$$

Proof of Theorem 4: Let $\tilde{P}$ be any general positive semi-definite symmetric matrix operator on $\tilde{V} = V_0 \times V_1 \times \ldots V_J$. We first prove: If $\tilde{Q}$ is any rectangular submatrix of $\tilde{P}$ belonging entirely to the lower triangle then

$$||\tilde{Q}||_{\tilde{V}\to\tilde{V}} \leq \frac{1}{2}\lambda_{\max}\ . \tag{39}$$

More precisely, if $\tilde{P} = ((T_{i,j} = T_{i|V_j}))$ and $\tilde{Q} = ((Q_{i,j}))$ with

$$Q_{i,j} = \begin{cases} T_{i,j} & 0 \leq j_0 \leq j \leq j_1 < i_0 \leq i \leq i_1 \leq J \\ 0 & \text{otherwise} \end{cases}$$

then, obviously,

$$\begin{aligned} ||\tilde{Q}||_{\tilde{V}\to\tilde{V}} &= \sup_{||\tilde{u}||,||\tilde{v}||\leq 1} \tilde{b}(\tilde{Q}\tilde{u},\tilde{v}) \\ &= \sup\left\{\tilde{b}(\tilde{P}\hat{u},\hat{v}) : \begin{array}{l} \hat{u} = (0,\ldots u_{j_0},\ldots,u_{j_1},0,\ldots,0), ||\hat{u}||_{\tilde{V}} \leq 1 \\ \hat{v} = (0,\ldots v_{i_0},\ldots,v_{i_1},0,\ldots,0), ||\hat{v}||_{\tilde{V}} \leq 1 \end{array}\right\}, \end{aligned}$$

c.f. also Figure 1.

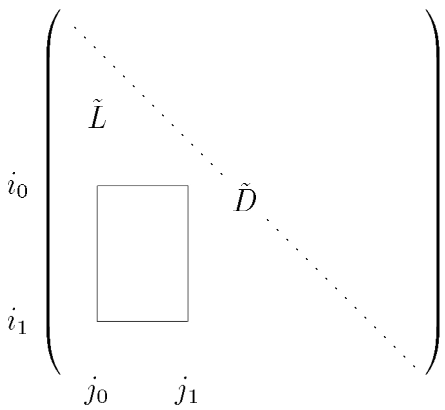



Figure 1: The submatrix $Q_{i,j}$ and its indices.

Since

$$\begin{aligned} \tilde{b}(\tilde{P}\hat{u},\hat{v}) &= \frac{1}{4}(\tilde{b}(\tilde{P}(\hat{u}+\hat{v}),(\hat{u}+\hat{v})) - \tilde{b}(\tilde{P}(\hat{u}-\hat{v}),(\hat{u}-\hat{v})) \\ &\leq \frac{1}{4}\lambda_{\max}\cdot||\hat{u}+\hat{v}||^2_{\tilde{V}}\ , \end{aligned}$$

$$\tilde{b}(\tilde{P}(\hat{u}+\hat{v}),(\hat{u}+\hat{v})) \leq \lambda_{\max}\cdot\|\hat{u}+\hat{v}\|^2_{\tilde{V}}, \quad \text{and} \quad 0 \leq \tilde{b}(\tilde{P}(\hat{u}-\hat{v}),(\hat{u}-\hat{v}))\ ,$$

we have

$$\tilde{b}(\tilde{P}\hat{u},\hat{v}) \leq \frac{1}{4}\lambda_{\max}\cdot\|\hat{u}+\hat{v}\|^2_{\tilde{V}}\ .$$

Furthermore,

$$\begin{aligned}\|\hat{u}+\hat{v}\|_{\tilde{V}}^2 &= \|(0,\ldots,0,u_{j_0},\ldots,u_{j_1},0,\ldots,0,v_{i_0},\ldots,v_{i_1},0,\ldots,0)\|_{\tilde{V}}^2 \\ &= \sum_{j=j_0}^{j_1} b_j(u_j,u_j)+\sum_{i=i_0}^{i_1} b_i(v_i,v_i)=\|\hat{u}\|_{\tilde{V}}^2+\|\hat{v}\|_{\tilde{V}}^2 \le 2\end{aligned}$$

and we obtain (39).

We now use induction with respect to $k$ to establish

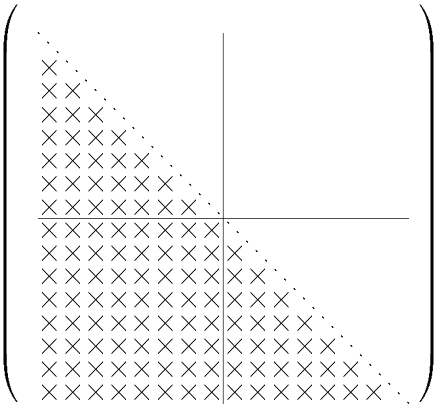
$$\|\tilde{L}\|_{\tilde{V}\to\tilde{V}} \le \frac{k}{2}\lambda_{\max}, \quad J=2^k-1, \quad k=1,2,3\ldots \tag{40}$$

from which the assertion of the Theorem 4 follows by monotonicity noting that $[\log_2(2J)]=k$ for $J=2^{k-1},\ldots,2^k-1$. For $k=1$, we have

$$\begin{pmatrix} 0 & 0 \\ T_{10} & 0 \end{pmatrix} = \tilde{L}$$

and we can apply (39).

Suppose now that $J=2^k-1$, and that for $J=2^{k-1}-1$ the result has already been proved. Split $\tilde{L}$ and analogously $\tilde{P}$ as in Figure 2.

$\tilde{L} =$ 

$$= \left(\begin{array}{c|c} \tilde{L}_1 & \tilde{0} \\ \hline \tilde{Q} & \tilde{L}_2 \end{array}\right) = \left(\begin{array}{c|c} 0 & 0 \\ \hline \tilde{Q} & 0 \end{array}\right) + \left(\begin{array}{c|c} \tilde{L}_1 & 0 \\ \hline 0 & \tilde{L}_2 \end{array}\right)$$

Figure 2: Splitting of $\tilde{L}$ for $J=15$.

By (39) we have

$$\|\tilde{Q}\|_{\tilde{V}\to\tilde{V}} \le \frac{1}{2}\lambda_{\max}\,.$$

For the block-diagonal matrix $\tilde{L}_1 \oplus \tilde{L}_2$ it is obvious that

$$\|\tilde{L}_1 \oplus \tilde{L}_2\|_{\tilde{V}\to\tilde{V}} = \max(\|\tilde{L}_1 \oplus \tilde{0}_2\|_{\tilde{V}\to\tilde{V}}, \|\tilde{0}_1 \oplus \tilde{L}_2\|_{\tilde{V}\to\tilde{V}})\,.$$

But to $\tilde{L}_1$, $\tilde{L}_2$ we can separately apply (40) with $k-1$. The operator $\tilde{P}_1$ corresponding to $\tilde{L}_1$ is

$$\tilde{P}_1 = \begin{pmatrix} T_{0,0} & T_{0,1} & \cdot\ \cdot & T_{0,2^{k-1}-1} \\ T_{1,0} & T_{1,1} & \cdot\ \cdot & T_{1,2^{k-1}-1} \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ T_{2^{k-1}-1,0} & T_{2^{k-1}-1,1} & \cdot\ \cdot & T_{2^{k-1}-1,2^{k-1}-1} \end{pmatrix}$$

and satisfies $\lambda_{\max}(\tilde{P}_1) \le \lambda_{\max}(\tilde{P})$ and $0 \le \lambda_{\min}(\tilde{P}) \le \lambda_{\min}(\tilde{P}_1)$. The operator $\tilde{P}_2$ corresponding to $\tilde{L}_2$ can be defined analogously. Thus,

$$\|\tilde{L}\|_{\tilde{V}\to\tilde{V}} \le \frac{1}{2}\lambda_{\max} + \frac{k-1}{2}\lambda_{\max} = \frac{k}{2}\lambda_{\max}$$

and the proof is completed. Now, the bound (37) follows directly from (31) and the bound (38) follows from (32) with (36), the relation $\omega_1 \le \lambda_{\max}$ and $\log_2(2J)+1 = \log_2(4J)$. $\Box$

**Remark**: One could have formulated a slightly more precise result:

$$\|\tilde{L}\|_{\tilde{V}\to\tilde{V}} \le \frac{1}{2}[\log_2(2J)] \cdot (\lambda_{\max}(\tilde{P}) - \lambda_{\min}(\tilde{P}))$$

which improves the above result in the case of non-overlapping subspaces, i.e. direct sums in (2). To this end, take suprema with respect to $\|\hat{u}\|_{\tilde{V}} = 1, \|\hat{v}\|_{\tilde{V}} = 1$ and replace at the respective places

$$\tilde{b}(\tilde{P}(\hat{u}-\hat{v}), (\hat{u}-\hat{v})) \ge \lambda_{\min}(\tilde{P}) \cdot \|\hat{u}-\hat{v}\|_{\tilde{V}}^2$$

and $\|\hat{u}+\hat{v}\|_{\tilde{V}}^2 = 2$. Compare [O1], where the asymptotical sharpness for $J \to \infty$ of the logarithmical factor in (36) and (37), respectively, has been studied.

# 5 Concluding remarks

As a whole, we have now a very compact basic convergence theory for the additive and multiplicative Schwarz methods, with assumptions for the additive variant which are in some sense minimal. The above results show very clearly the crucial role played by the values $\lambda_{\max}$ and $\lambda_{\min}$ of a splitting. A survey of some basic splittings, suitable for finite element discretizations of variational problems in Sobolev spaces, is given in [O2], [O3].

Theorem 4 is of theoretical importance: It shows that for *any* variational problem and *any* additive subspace splitting (including an arbitrary choice of the auxiliary forms

$b_j(\cdot,\cdot)$), there exists a proper scaling of the multiplicative Schwarz algorithm such that the convergence rate depends only on the condition number of the additive Schwarz operator, and logarithmically on the number of subspaces. *Good preconditioning via the additive Schwarz formulation implies fast convergence of the multiplicative Schwarz algorithm* ! It is an open question for which classes of matrices $\tilde{P}$ it will be possible to eliminate the logarithmic factor in Theorem 4. Note, however, that for general matrices $\tilde{P}$, this cannot be achieved; c.f. the examples given in [KP] and [O1].

We note that, as a rule, the convergence rate of the multiplicative algorithm with an *optimal* scaling factor $\omega$ is better than indicated by the above estimates. This makes it necessary to develop much more elaborate techniques for the multiplicative case. One approach is the use of strengthened Cauchy-Schwarz inequalities, and the selection of suitable auxiliary subspace decompositions

$$V = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_J \ , \quad \mathcal{V}_j \subset V_j \ , \ j = 0, 1, \ldots, J$$

which may lead to improved estimates. For details, we refer to [Ys], [X]. In particular, let us state the estimate

$$\rho_{ms} \leq \sqrt{1 - \frac{\hat{\lambda}_{\min} \cdot (\frac{2}{\omega} - \omega_1)}{(\frac{1}{\omega} + \sqrt{\omega_1 J \hat{\lambda}_{\max}})^2}}$$

of the convergence rate of the multiplicative method given in Theorem 5.4 of [Ys]. Here $\hat{\lambda}_{\min}$ and $\hat{\lambda}_{\max}$ are the minimal and maximal eigenvalues of the additive Schwarz operator corresponding to the auxiliary splitting $V = \bigoplus \mathcal{V}_j$ (the forms on $\mathcal{V}_j$ are inherited from $V_j$). Compared to our above result (37), we see that the dependence on $J$ is much worse.

The effect of individual scaling (this is replacing $\omega$ by $\omega_j \ , \ j = 0, 1, \ldots, J$) on the convergence rate in the multiplicative algorithm and its influence on the spectral bounds of the additive Schwarz operator (i.e. replacing $b_j(\cdot,\cdot)$ by $b_j(\cdot,\cdot)/\omega_j$) has not yet been studied in a satisfactory way. Note that constant scaling ($\omega_j = \omega$, $\forall j = 0, \ldots, J$) does not change the condition of the additive operator though it may substantially influence the behavior of the multiplicative algorithm. Also, the effect of the ordering of the subspaces on the convergence rate of the multiplicative method is not yet well understood.

We conclude with a few remarks.

- According to Theorems 1, 2, and 3, the analysis of Schwarz algorithms for a *given* splitting may be reduced to the proof, with best possible constants $\lambda_{\max}$ and $\lambda_{\min}$, of the norm equivalence

$$a(u,u) \approx |||u|||^2 = \inf_{u_j \in V_j \, : \, u = \sum_{j=0}^{J} u_j} \sum_{j=0}^{J} b_j(u_j, u_j) \ .$$

Thus, the basic question is to choose the splitting and the approximate solvers which correspond to the auxiliary bilinear forms $b_j(\cdot,\cdot)$ on the subspaces in such a way that, on one hand, good values for $\lambda_{\max}$ and $\lambda_{\min}$ can be derived and, on the other hand, an overall algorithm with cheap components will be obtained. What is cheap strongly depend on the computer environment available and the problem classes under consideration. Furthermore, there are the questions of data structures, serial versus vectorized and parallel algorithms, respectively, and adaptivity. Nevertheless, in view of the above theory, it is good to have many potential candidates of splittings for a given problem class, with a precise knowledge on the corresponding $\lambda_{\max}$ and $\lambda_{\min}$ before one starts to design efficient practical solvers in a given environment.

- If one has already found a 'good' splitting one may obtain others by simple procedures such as *refinement*, *clustering*, and, under certain restrictions, *selection*. Refinement means that some or all $V_j$ are replaced in turn by splittings $V_j = \sum_i V_{j,i}$ with possibly other auxiliary forms $b_{j,i}(\cdot,\cdot)$ on these new subspaces. Clustering is the inverse process - some of the subspaces in the splitting are grouped together. The whole theory of such techniques can be described by the obvious inequalities (cf. (4) and (5))

$$\begin{aligned}
\kappa(\{V;a\};\sum_{j,i}\{V_{j,i},b_{j,i}\}) &\leq \frac{\Lambda_1}{\Lambda_0}\cdot\kappa(\{V;a\};\sum_j\{V_j,b_j\}) \\
\kappa(\{V;a\};\sum_j\{V_j,b_j\}) &\leq \frac{\Lambda_1}{\Lambda_0}\cdot\kappa(\{V;a\};\sum_{j,i}\{V_{j,i},b_{j,i}\})
\end{aligned}$$

where

$$\Lambda_0 \equiv \min_{j=0,\ldots,J}\{\lambda_{\min}(P_j)\} \qquad \Lambda_1 \equiv \max_{j=0,\ldots,J}\{\lambda_{\max}(P_j)\}$$

and $P_j : V_j \to V_j$, denotes the additive Schwarz operator corresponding to the subsplitting of $V_j$.

Selection can be used, e.g., for dealing with certain adaptive algorithms. Starting from a given decomposition (2), one may ask for properties of the splitting

$$\hat{V} = \sum_{j=0}^{J}\hat{V}_j\ ; \quad \hat{V}_j \subset V_j$$

for the new (selected) subspace $\hat{V} \subset V$, where the bilinear forms on $\hat{V}$ and $\hat{V}_j$, respectively, are inherited from $V$ and $V_j$, respectively. Note that the choice $\hat{V}_j = \{0\}$ is also allowed. It is obvious that with $\hat{P} := \sum_{j=0}^{J}\hat{T}_j$, i.e. that for the additive Schwarz operator $\hat{P}$ associated to the new splitting, the relation

$$\lambda_{\max}(\hat{P}) \leq \lambda_{\max}$$

holds for the selection. If (2) is a direct sum of subspaces then an analogous but opposite estimate also holds for the minimal eigenvalues. For overlapping decompositions this is in general not true, and one has to carefully give an estimation of $\lambda_{\min}(\hat{P})$ from below for each particular selection of subspaces $\hat{V}_j \subset V_j$, $j = 0, 1, , \ldots, J$.

- The use of the fictitious space lemma opens the way for dealing with external approximation schemes (e.g., such as arising from nonconforming discretizations) in the same conceptional way. Thus, it is not really necessary to explicitly split $V$ into *subspaces*. Instead, given any collection of Hilbert spaces $V_j$ equipped with their own scalar products $(\cdot,\cdot)_j$ and auxiliary s.p.d. forms $b_j(u_j, v_j) = (B_j u_j, v_j)_j$ $(u_j, v_j \in V_j)$, it suffices to define a surjective linear mapping

$$R \,:\, V_0 \times \ldots \times V_J \to V \qquad (R\tilde{u} = \sum_{j=0}^{J} R_j u_j)$$

and to check the constants in the two-sided norm comparison

$$a(u,u) \approx \inf_{v_j \in V_j \,:\, u = \sum_{j=0}^{J} R_j v_j} \sum_{j=0}^{J} b_j(v_j, v_j) \qquad \forall\, u \in V\ .$$

The preconditioner for the additive Schwarz algorithm now takes the form

$$C_{\sum V_j} = \sum_{j=0}^{J} R_j B_j^{-1} R_j^*,$$

with $R_j^* \,:\, V \to V_j$ defined by $(R_j u_j, v) = (u_j, R_j^* v)_j \quad \forall u_j \in V_j\,,\, v \in V,\ j = 0, \ldots, J$.

Formally, the mapping $R$ transfers the external construction $\{V_j, b_j\}$ into an *internal subspace* splitting

$$V = \sum_{j=0}^{J} R_j V_j \equiv \sum_{j=0}^{J} \hat{V}_j\ , \qquad \hat{b}_j(\hat{u}_j, \hat{v}_j) = b_j(u_j, v_j)$$

where, for any $\hat{u}_j \in R_j V_j$, we denote by $u_j \in V_j$ the unique element satisfying $R_j u_j = \hat{u}_j$ and $b_j(u_j, \bar{u}_j) = 0$ for all $\bar{u}_j \in \mathrm{Ker}\,(R_j)$. This shows the role of the $R_j$, and it is clear that a proper choice is crucial for the success; c.f. also [BPX], [D].

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# References


[BM] P.E.Bjørstad, J.Mandel, On the spectra of orthogonal projections with applications to parallel computing. BIT 31 (1991) 76-88.

[BP] J.H.Bramble, J.E.Pasciak, New estimates for multilevel algorithms including the V-cycle. Math. Comp. 60 (1993), 447-471.

[BPWX1] J.H.Bramble, J.E.Pasciak, J.Wang, J.Xu, Convergence estimates for product iterative methods with applications to domain decomposition. Math. Comp. 57 (1991), 1-21.

[BPWX2] J.H.Bramble, J.E.Pasciak, J.Wang, J.Xu, Convergence estimates for multigrid algorithms without regularity assumptions. Math. Comp. 57 (1991) 23-45.

[BPX] J.H.Bramble, J.E.Pasciak, J.Xu, The analysis of multigrid algorithms with nonested spaces or noninherited quadratic forms. Math. Comp. 56 (1991), 1-34.

[D] W.Dörfler, Hierarchical bases for elliptic problems. Math. Comp. 58 (1992) 513-529.

[DW1] M.Dryja, O.Widlund, Some domain decomposition algorithms for elliptic problems. in: Iterative Methods for Large Linear Systems, San Diego, California, L.Hayes, D.Kincaid (eds.), 273-291, Acad. Press, 1989.

[DW2] M.Dryja, O.Widlund, Towards a unified theory of domain decomposition algorithms for elliptic problems. in: Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, Houston, Texas, March 20-22 1989, T.F.Chan, R.Glowinski, J.Périaux, O.Widlund (eds.), SIAM, 1990.

[DW3] M.Dryja, O.Widlund, Additive Schwarz methods for elliptic finite element problems in three dimensions. in: Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations, T.F. Chan, D.E. Keyes, G.A. Meurant, J.S. Scroggs, R.G. Voigt (eds.), SIAM, 1992.

[DSW] M.Dryja, B.F.Smith, O.Widlund, Schwarz analysis of iterative substructuring algorithms for elliptic problems in three dimensions. Preprint, Courant Inst., New York Univ., May 1993, to appear in SIAM J. Numer. Anal.

[G1] M.Griebel, Multilevel algorithms considered as iterative methods on semidefinite systems. SIAM J. Sci. Comp. 15(3) (1994), 547-565.

[G2] M.Griebel, Multilevelmethoden als Iterationsverfahren über Erzeugendensystemen. Teubner Skripten zur Numerik, Teubner Verlag, Stuttgart, 1994.

[GO1] M.Griebel, P.Oswald, On additive Schwarz preconditioners for sparse grid discretizations. Numer. Math. 66 (1994), 449-464.

[GO2] M.Griebel, P.Oswald, Remarks on the abstract theory of additive and multiplicative Schwarz methods. Report TUM-I9314, TU München, 1993.

[H] W.Hackbusch, Iterative Lösung großer schwachbesetzter Gleichungssysteme, 2. Auflage, Teubner, Stuttgart, 1993.

[KP] S.Kwapien, A.Pelczynski, The main triangle projection in matrix spaces and its applications. Studia Math. 34 (1970) 43-68.

[MN] A.M.Matsokin, S.V.Nepomnyaschikh, Schwarz alternating method in a subspace. Sov. Math. (Izv. vuz.) 29 (1985) 78-84.

[N1] S.V.Nepomnyaschikh, Decomposition and fictitious domain methods for elliptic boundary value problems. in: Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations, T.F. Chan, D.E. Keyes, G.A. Meurant, J.S. Scroggs, R.G. Voigt (eds.), SIAM, 1992.

[N2] S.V.Nepomnyaschikh, Mesh theorems on traces, normalization of function traces and their inversion. Sov. J. Numer. Anal. Math. Modelling 6 (3) (1991), 223-242.

[O1] P.Oswald, On the convergence rate of SOR : A worst case estimate. Computing 52 (3) (1994), 245-255.

[O2] P.Oswald, Multilevel Finite Element Approximation. Theory & Applications. Teubner Skripten zur Numerik, Teubner Verlag, Stuttgart, 1994.

[O3] P.Oswald, Stable splittings of Sobolev spaces and applications. Preprint Math/93/5, FSU Jena, 1993.

[W] O.Widlund, Some Schwarz methods for symmetric and nonsymmetric elliptic problems. in: Fifth International Symposium on Domain Decomposition Methods for Partial Differential Equations, T.F. Chan, D.E. Keyes, G.A. Meurant, J.S. Scroggs, R.G. Voigt (eds.), SIAM, 1992.

[X] J.Xu, Iterative methods by space decomposition and subspace correction. SIAM Review 34 (1992) 581-613.

[Yo] D.M.Young, Iterative solutions of large linear systems. Academic Press, New York, 1971.

[Ys] H.Yserentant, Old and new convergence proofs for multigrid methods. Acta Numerica 1993, Cambridge Univ. Press, N. Y., 1993, 285-326.

[Z] X.Zhang, Multilevel Schwarz methods. Numer. Math. 63 (1992) 521-539.