On additive Schwarz preconditioners for sparse grid discretizations

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Summary. Based on the framework of subspace splitting and the additive Schwarz scheme, we give bounds for the condition number of multilevel preconditioners for sparse grid discretizations of elliptic model problems. For a BXP-like preconditioner we derive an estimate of the optimal order O(1) and for a HB-like variant we obtain an estimate of the order $O(k^2 \cdot 2^{k/2})$, where k denotes the number of levels employed. Furthermore, we confirm these results by numerically computed condition numbers.

Contents :

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1. Introduction

Recently, sparse grids have been brought into discussion by Zenger [Z] as a new discretization technique for the approximate solution of elliptic boundary value problems. To explain the basic idea which has been used previously under different names for other purposes (trigonometric and spline approximation theory, optimal recovery and quadratures, see e.g. [DS, T1, T2]), let us consider the Poisson equation with homogeneous Dirichlet boundary conditions

$$-\Delta u = f \quad , \qquad u \in H_0^1(\Omega) \, ,$$

in the unit square $\Omega = [0,1]^2$. The traditional (full grid) approach to solving this problem numerically by the finite element method is to equip Ω with a partition into squares of sidelength h, to "project" the variational problem onto the corresponding subspace of bilinear finite element functions in $H_0^1(\Omega)$, and to solve the resulting linear system of dimension $O(h^{-2})$ (for convenience, from now on we will be restricted to the case $h = 2^{-k}\,,\ k \geq 1$, and write $\mathcal{R}_{k,k}$ and $\hat{V}_{k,0} = V_{k,k,0}$ for the partitions and finite element subspaces). If $u \in H^2(\Omega)$ then the obtained approximate solution satisfies an $O(h)$ error estimate in the energy norm and an $O(h^2)$ error estimate in the L_2-norm, respectively. Thus, high accuracy requires the solution of very large linear systems. This situation gets still worse in the 3D case.

The observation which leads to the sparse grid approach is that, under additional regularity assumptions on higher order mixed derivatives of u, almost the same approximation rates can be achieved by a subspace of dimension $O(h^{-1} \cdot \log h^{-1})$, namely by the sparse grid space

$$\tilde{V}_{k,0} = V_{1,k-1,0} + V_{2,k-2,0} + \ldots + V_{k-1,1,0}$$

where $V_{k_1,k_2,0}$ denotes the subspaces of bilinear finite element functions in $H_0^1(\Omega)$ with respect to the partition $\mathcal{R}_{k_1,k_2}$ of Ω into rectangles of size 2^{-k_1} by 2^{-k_2}, $k_1, k_2 \geq 1$, see [Z] for more details, and [T1] for the trigonometric counterpart. Examples of sparse grids are given in Figure 1.

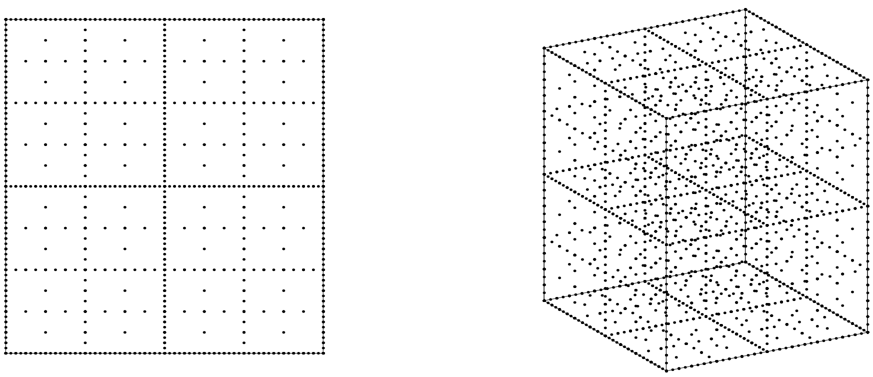

Figure 1: Nodal points of the 2D sparse grid and the 3D sparse grid, $k = 7$.

To be more precise, let the solution u be sufficiently smooth to satisfy

$$|\frac{\partial u^4}{\partial x^2 \partial y^2}| \leq C.$$

Then, it can be shown, that, with respect to the energy norm, the approximation error of the sparse grid solution is of the order $O(h)$ like in the full grid approach. Regarding the L_2-norm, a slight deterioration from $O(h^2)$ to $O(h^2 \cdot \log h^{-1})$ can be observed. For details, see [B]. Though the theoretical justification of the error estimates and the behaviour of the method when applied to realistic problems in more complicated domains is still incomplete, the numerical testing reported in [Z, GSZ, G1, G2, B] definitively shows the capabilities of this dimension reduction. Especially for 3D problems, the sparse grid technique will certainly prove its practical importance in the near future.

The present paper contributes to the justification of fast iterative solvers for the linear systems arising from sparse grid discretizations. Until now, for this purpose different strategies have been tested numerically, cf. [GSZ, G1, G2, B]. Note that the starting point was a sparse hierarchical basis approach [Z]. Since for the hierarchical basis method of Yserentant [Y] and for the more recent multilevel additive Schwarz preconditioners like the BPX scheme [BPX, O1, X, Zh] and related methods [H] the good performance for full grid discretizations has been shown theoretically, it is natural to ask for analogous results in the sparse grid case.

The main result of this paper is a new and easily implementable BPX-like multilevel preconditioner for sparse grid discretizations of H^1 elliptic boundary value problems on the unit square which possesses $O(1)$ asymptotics for the condition numbers (Section 4). Moreover, in Section 5 almost final results are obtained for the sparse hierarchical basis method [Z, L, B]. These theoretical results are supported by numerical tests. Though the main results are stated for two space dimensions, the proofs given apply to the higher-dimensional case as well, see Section 6. Our methods are close to those previously developed in [O1, O2, O3], and use the abstract theory of additive Schwarz schemes as the appropriate framework, see Sections 2 and 3.

2. Abstract additive Schwarz schemes and subspace splittings

In this Section we give a short summary of results of several authors (see, e.g., [DW, W, X, Zh, BM, O1]) on the theory of additive Schwarz schemes for the solution of symmetric variational problems in finite-dimensional Hilbert spaces.

Our aim is to derive estimates of the condition number for the additive Schwarz operator that is associated to the variational problem

$$\text{find } u \in V: \quad a(u,v) = \Phi(v) , \quad \forall v \in V , \tag{1}$$

and to a subspace splitting

$$V = \sum_{i=1}^{N} V_i \tag{2}$$

with $N > 1$ a given natural number.

Our precise assumptions are as follows. Suppose that V is a finite-dimensional Hilbert space with inner product $(\cdot,\cdot)_V$ and norm $\|\cdot\|_V$. Suppose further that the subspaces $V_i \subset V$ that form the additive splitting (2) are also Hilbert spaces with their own inner products $(\cdot,\cdot)_{V_i}$ and corresponding norms $\|\cdot\|_{V_i}$, $i = 1,\ldots,N$.. Let $a(\cdot,\cdot)$ be a symmetric V-elliptic bilinear form on V that satisfies the two-sided inequality

$$c_a \cdot \|u\|_V^2 \le \|u\|_E^2 \equiv a(u,u) \le C_a \cdot \|u\|_V^2 \ , \quad \forall u \in V \ , \tag{3}$$

with positive constants $0 < c_a \le C_a < \infty$. Moreover, let Φ be a bounded linear functional on V.

We introduce auxiliary symmetric V_i-elliptic bilinear forms $b_i(\cdot,\cdot)$ on V_i where

$$c_b \cdot \|u_i\|_{V_i}^2 \le b_i(u_i,u_i) \le C_b \cdot \|u_i\|_{V_i}^2 \ , \quad \forall u_i \in V_i \ , \tag{4}$$

holds uniformly in $i = 1,\ldots,N$ with two positive constants $0 < c_b \le C_b < \infty$.

Now, we introduce

$$|||u|||_V = \inf \left\{ \left(\sum_{i=1}^N \|u_i\|_{V_i}^2 \right)^{1/2} \ : \ u = \sum_{i=1}^N u_i \ , \ u_i \in V_i \ , \ i = 1,\ldots,N \right\} \ , \tag{5}$$

that defines an equivalent norm on V, i.e. it holds

$$c \cdot \|u\|_V^2 \le |||u|||_V^2 \le C \cdot \|u\|_V^2 \ , \quad \forall u \in V \ , \tag{6}$$

with constants $0 < c \le C < \infty$.

We call $\kappa(V,\{V_i\}) = \inf C/c$ (where the infimum is taken in (6) with respect to all possible constants c, C) the *stability constant* of the splitting (2). Note that $\kappa(V,\{V_j\})$ depends strongly on the inner products $(.,.)_V$ and $(.,.)_{V_i}$ associated to V and V_i, respectively. In the special case of $(\cdot,\cdot)_V \equiv a(\cdot,\cdot)$ and $(\cdot,\cdot)_{V_i} \equiv b_i(\cdot,\cdot)$, $i = 1,\ldots,N$, we write $\kappa(a,\{b_i\})$ for this quantity. As a trivial result we have

Lemma 1. *With the above notation, we obtain the estimate*

$$\kappa(a,\{b_i\}) \le C_a/c_a \cdot C_b/c_b \cdot \kappa(V,\{V_i\}) \quad . \tag{7}$$

Now, let the operator $P_{V_i} : V \to V_i$ be defined by the variational problem

$$b_i(P_{V_i}u, v_i) = a(u, v_i) \ , \quad \forall v_i \in V_i \ . \tag{8}$$

Furthermore, let $\phi_i \in V_i$ be defined by

$$b_i(\phi_i, v_i) = \Phi(v_i) \ , \quad \forall v_i \in V_i \ , \tag{8'}$$

where $i = 1,\ldots,N$. Then, the additive Schwarz operator P_V is given by

$$P_V = \sum_{i=1}^N P_{V_i} \ : \ V \to V \tag{9}$$

and the associated right hand side ϕ is given by

$$(9') \qquad \phi = \sum_{i=1}^{N} \phi_i \in V \ ,$$

Altogether, we have

Lemma 2. (i) *The variational problem* (1) *is equivalent to solving the operator equation*

$$(10) \qquad P_V u = \phi \quad , \quad u \in V \quad .$$

(ii) *The operator P_V is symmetric positive definite (with respect to the inner product $a(\cdot,\cdot)$) and its spectral condition number is given by*

$$(11) \qquad \kappa(P_V) \equiv \frac{\lambda_{\max}(P_V)}{\lambda_{\min}(P_V)} = \kappa(a, \{b_i\}) \quad .$$

Proof. This lemma is implicitly stated in [W], [Z] and has many contributors. We give the elementary proof for the sake of completeness, only.

We start with (ii). For arbitrary $u, v \in V$, we have

$$\begin{aligned} a(P_V u, v) &= \sum_{i=1}^{N} a(P_{V_i} u, v) = \sum_{i=1}^{N} a(v, P_{V_i} u) \\ &= \sum_{i=1}^{N} b_i(P_{V_i} v, P_{V_i} v) = \sum_{i=1}^{N} b_i(P_{V_i} u, P_{V_i} v) \\ &= \sum_{i=1}^{N} a_i(u, P_{V_i} v) = a(u, P_V v) \ . \end{aligned}$$

Moreover, we obtain

$$a(P_V u, u) = \sum_{i=1}^{N} b_i(P_{V_i} u, P_{V_i} u) \geq 0 \quad , \qquad u \in V \ .$$

Equality is reached if and only if $P_{V_i} u = 0$ or, equivalently, if u is a-orthogonal to each V_i , $i = 1, \ldots, N$, which gives $u = 0$.

Thus, P_V is symmetric positive definite and invertible. Assertion (i) is now obvious (see (8)-(8') and (9)-(9')). The formula for the spectral condition number follows from the identity

$$a(P_V^{-1} u, u) = \inf_{u_i \in V_i \,:\, u = \sum_i u_i} \sum_{i=1}^{N} b_i(u_i, u_i) \quad , \qquad u \in V \ .$$

Indeed, let $u = \sum_{i=1}^{N} u_i$, $u_i \in V_i$, $i = 1, \ldots, N$, be an arbitrary decomposition of $u \in V$. Then,

$$\begin{aligned} a(P_V^{-1} u, u) &= \sum_{i=1}^{N} a(P_V^{-1} u, u_i) = \sum_{i=1}^{N} b_i(\underbrace{P_{V_i} P_V^{-1} u}_{=w_i}, u_i) \\ &\leq \left(\sum_{i=1}^{N} b_i(w_i, w_i) \right)^{1/2} \cdot \left(\sum_{i=1}^{N} b_i(u_i, u_i) \right)^{1/2} \end{aligned}$$

where

$$\sum_{i=1}^{N} b_i(w_i, w_i) = \sum_{i=1}^{N} a(P_V^{-1}u, w_i) = a(P_V^{-1}u, u) \ .$$

Thus, we have

$$a(P_V^{-1}u, u) \le \sum_{i=1}^{N} b_i(u_i, u_i)$$

for any decomposition of $u \in V$. Equality is obtained for $u = \sum_i w_i$.

Lemmas 1 and 2 imply the following strategy for the design of a solver for (1) using the additive Schwarz formulation (10) :

Step 1. Given the variational problem (1), choose $(\cdot,\cdot)_V$ such that C_a/c_a is moderate (cf. (3)).

Step 2. Choose V_i and $(\cdot,\cdot)_{V_i}$, $i = 1, \ldots, N$, such that the stability constant $\kappa(V, \{V_i\})$ of the corresponding splitting (2) is moderate.

Step 3. Choose the auxiliary symmetric forms b_i on V_i , $i = 1, \ldots, N$, such that C_b/c_b is moderate (cf. (4)), and that the variational problems (8)-(8') can be solved with substantially less effort than (1).

Step 4. Solve (10) by the conjugate gradient method (or by any other simple iteration method which converges rapidly for well–conditioned operator equations).

The crucial point of the theoretical analysis of almost any existing additive Schwarz scheme (domain decomposition methods and multilevel additive Schwarz methods) is an appropriate result in Step 2 of the above strategy. With respect to multilevel additive Schwarz methods, the norm equivalencies (6) that are necessary for discretized variational problems (1) in Sobolev spaces have a nice background in approximation theory. This background will be used below in order to develop optimal (or suboptimal) additive Schwarz preconditioners for sparse grid discretizations of elliptic boundary value problems.

3. Stable splittings for sparse grid discretizations.

Throughout this paper, let $\Omega \equiv [0,1]^d$ be the d-dimensional cube ($d \ge 2$). For given $\mathbf{k} \in Z_+^d$, $\mathbf{k} \equiv (k_1, \ldots, k_d)$, let $\mathcal{R}_\mathbf{k}$ be the tensor-product partition with uniform stepsize 2^{-k_j} into the j-th coordinate direction, $j = 1, \ldots, d$. By $V_\mathbf{k}$ we denote the spaces of multilinear finite element functions with respect to $\mathcal{R}_\mathbf{k}$, $\mathbf{k} \in Z_+^d$; if homogeneous boundary conditions are required, we also consider $V_{\mathbf{k},0} = V_\mathbf{k} \cap H_0^1(\Omega)$ (here, $\mathbf{k} \ge \mathbf{1} \equiv (1, \ldots, 1)$ is natural). Note that higher order C^0-elements, C^1-element spaces for treating fourth order problems, and slightly more complicated boundary conditions may be considered as well. However, we restrict our attention to the most simple case. These spaces satisfy the monotonicity property

$$V_{\mathbf{k}} \subset V_{\mathbf{m}} \ , \quad V_{\mathbf{k},0} \subset V_{\mathbf{m},0} \quad , \qquad \mathbf{k} \le \mathbf{m}. \tag{12}$$

Now we introduce, for $k \in Z_+$, the full grid spaces

$$\hat{V}_k = V_{(k,\ldots,k)} \quad \text{resp.} \quad \hat{V}_{k,0} = V_{(k,\ldots,k),0} \ (k > 0)$$

and the sparse grid spaces

$$\tilde{V}_k = \sum_{|\mathbf{k}|_1 \le k} V_{\mathbf{k}} \quad \text{resp.} \quad \tilde{V}_{k,0} = \sum_{|\mathbf{k}|_1 \le k} V_{\mathbf{k},0} \ (k \ge d) \ .$$

(here and in the following we denote $|\mathbf{k}|_1 = \sum_j k_j$ and $|\mathbf{k}|_\infty = \max_j k_j$ whenever $\mathbf{k} \in Z_+^d$).

The properties of sequences of full grid spaces are well understood. We will make use of the following basic stability result which is applicable to symmetric second order elliptic boundary value problems . Let

$$(u, v)_{H^1} = (u, v)_{L_2} + (\nabla u, \nabla v)_{L_2} \quad , \qquad u, v \in H^1(\Omega) \ ,$$

where $(\cdot, \cdot)_{L_2}$ denotes the usual inner product in $L_2(\Omega)$. On $H_0^1(\Omega)$, an equivalent inner product is introduced by $(u, v)_{H_0^1} = (\nabla u, \nabla v)_{L_2}$.

Lemma 3. *For arbitrarily fixed $k \in Z_+$, let $V = \hat{V}_k$ resp. $V = \hat{V}_{k,0}$ $(k \ge 1)$ be equipped with the H^1 resp. H_0^1 inner product introduced above. Consider the additive splitting of V into its subspaces $V_i = \hat{V}_i$, $i = 0, \ldots, k$, resp. $V_i = \hat{V}_{i,0}$, $i = 1, \ldots, k$ where in both cases the inner product on V_i is given by the scaled L_2 scalar product $2^{2i}(\cdot, \cdot)_{L_2}$. Then, the stability constants of these splittings remain bounded independently of k :*

$$\kappa(V, \{V_i\}) \le \kappa_\infty < \infty \qquad \forall \ k \ . \tag{13}$$

As shown in [O1, O2], Lemma 3 immediately follows from a result of [O3] on *infinite* splittings of $H^1(\Omega)$ resp. $H_0^1(\Omega)$ with respect to the corresponding sequences of finite element subspaces. The latter result is a consequence of the norm equivalence

$$\|u\|_{H^1}^2 \approx \|u\|_{L_2}^2 + \sum_{i=0}^{\infty} 2^{2i} \cdot E_i(u)_{L_2}^2 \quad (E_i(u)_{L_2} = \inf_{v_i \in \hat{V}_i} \|u - v_i\|_{L_2}) \tag{14}$$

for $u \in H^1(\Omega)$ that is proved, as a very particular case, in e.g. [O3, O4, DP] (see also [BY]).

Lemma 3 is the starting point for the main result of this Section. We will give the details for the H^1 situation only, the modifications for the H_0^1 case are quite obvious. Fix some $k \in Z_+$, consider (as V) the space $\tilde{V}_k$ equipped with the H^1 inner product, and denote for $l = 0, \ldots, k$

$$\tilde{V}_{k,l} = \tilde{V}_k \cap \hat{V}_l \quad , \quad (\cdot, \cdot)_{\tilde{V}_{k,l}} = 2^{2l} \cdot (\cdot, \cdot)_{L_2}$$

as well as

$$V_{k,l}^* = \bigcup_{\mathbf{m}: |m|_1 \le k, |m|_\infty = l} V_{\mathbf{m}} \quad , \quad (\cdot, \cdot)_{V_{k,l}^*} = 2^{2l} \cdot (\cdot, \cdot)_{L_2} \ .$$

Obviously,

$$V^*_{k,l} \subset \tilde{V}_{k,l} \subset \hat{V}_l \qquad (l \leq k) \tag{15}$$

and

$$V^*_{k,l} = \tilde{V}_{k,l} = \hat{V}_l \qquad (l \cdot d \leq k)\,. \tag{15'}$$

Theorem 1. *The splittings*

$$\tilde{V}_k = \sum_{l=0}^{k} \tilde{V}_{k,l} \quad \textit{resp.} \qquad \tilde{V}_k = \sum_{l=0}^{k} V^*_{k,l} \tag{16}$$

possess stability constants that are bounded uniformly in k, i.e.

$$\kappa(\tilde{V}_k, \{\tilde{V}_{k,l}\}) \leq \tilde{\kappa}_\infty < \infty \quad \textit{resp.} \quad \kappa(\tilde{V}_k, \{V^*_{k,l}\}) \leq \kappa^*_\infty < \infty \quad \forall\, k\,. \tag{17}$$

Proof. Let $\{f_r\,,\, r \geq 0\}$, denote the L_2-orthonormal Franklin system with respect to $[0,1]$, i.e. the system obtained from the Faber-Schauder system by the Schmidt orthogonalization procedure (note that the Franklin system was the first historical example of an *orthonormal Schauder basis* in $C[0,1]$ while the Faber-Schauder system was the first example of a Schauder basis in this space, see [KS] for more details). Define

$$f_{\mathbf{r}}(x) = f_{r_1}(x_1) \cdot \ldots \cdot f_{r_d}(x_d) \quad , \quad x \equiv (x_1, \ldots, x_d)\,,\; \mathbf{r} \in Z^d_+$$

and

$$W_{\mathbf{m}} = \; \mathrm{span}\{f_{\mathbf{r}} : 2^{m_j-1} < r_j \leq 2^{m_j}\,,\, j = 1, \ldots, d\} \quad , \quad \mathbf{m} \in Z^d_+ \quad (2^{-1} \equiv -1)\,.$$

We list some obvious properties of these objects which exclusively follow from the $L_2(0,1)$-orthogonality of the Franklin functions and the fact that span $\{f_r : r \leq 2^m\}$ coincides with the space of linear splines with respect to the uniform partition of $[0,1]$ into 2^m intervals :

$$\{f_{\mathbf{r}} : \mathbf{r} \in Z^d_+\} \quad \text{is an orthonormal system in } L_2(\Omega),$$

$$V_{\mathbf{k}} = \bigoplus_{\mathbf{m} \leq \mathbf{k}} W_{\mathbf{m}} \quad , \quad \forall\, \mathbf{k} \in Z^d_+\,, \tag{18}$$

and, as a consequence of (18), also

$$\tilde{V}_{k,l} = \bigoplus_{|\mathbf{m}|_1 \leq k, |\mathbf{m}|_\infty \leq l} W_{\mathbf{m}} \quad , \quad l = 0, \ldots, k\,. \tag{19}$$

Now we turn to the proof of the required two-sided norm estimates (cf. (6)). Due to (15), the lower estimates (6) follow for both splittings from the corresponding lower estimate for the splitting considered in Lemma 1 in the full grid case. To establish the upper estimate in (6), we have to construct a "good" decomposition with respect to the given subspace splitting. To this end, we use the orthogonal decomposition of $u \in \tilde{V}_k$ with respect to the multivariate Franklin system, and denote the block corresponding to $W_{\mathbf{m}}$ by $w_{\mathbf{m}}$. Then

$$u = \sum_{|\mathbf{m}|_1 \le k} w_{\mathbf{m}} = \sum_{l=0}^{k} \sum_{|\mathbf{m}|_1 \le k, |\mathbf{m}|_\infty = l} w_{\mathbf{m}} \equiv \sum_{l=0}^{k} u^*_{k,l}$$

is a "good" decomposition of u into $u^*_{k,l} \in V^*_{k,l} \subset \tilde{V}_{k,l}$. Indeed, by the L_2 orthogonality properties, (18), and the definition of the best approximations given in (14), we have

$$\begin{aligned} \|u^*_{k,l}\|^2_{L_2} &= \sum_{|\mathbf{m}|_1 \le k, |\mathbf{m}|_\infty = l} \|w_{\mathbf{m}}\|^2_{L_2} \\ &\le \sum_{|\mathbf{m}|_1 \le k, |\mathbf{m}|_\infty \ge l} \|w_{\mathbf{m}}\|^2_{L_2} = E_{l-1}(u)^2_{L_2} , \end{aligned}$$

with $E_{-1}(u)_{L_2} \equiv \|u\|_{L_2}$ if $l = 0$. It remains to sum up with the forefactors 2^{2l}, and to apply (14). This proves the Theorem.

Remark 1. It can be seen easily that the above proof works for any V of the form

$$V \equiv V_\Gamma = \sum_{\mathbf{m} \in \Gamma} V_{\mathbf{m}} \quad ,$$

where $\Gamma \subset Z^d_+$ is an arbitrarily fixed finite set (without loss of generality, we may assume that $\mathbf{m} \in \Gamma$ implies $\mathbf{m}' \in \Gamma$ for all $\mathbf{m}' \le \mathbf{m}$). Instead of $V^*_{k,l}$, one has to use

$$V^*_{\Gamma,l} = \sum_{\mathbf{m} \in \Gamma : |\mathbf{m}|_\infty = l} V_{\mathbf{m}},$$

with the scaled L_2 scalar products as above. The stability constant is bounded from above uniformly in Γ.

In particular, this observation yields splittings for any $V = V_{\mathbf{k}}$ into smaller spaces of the same type, with stability constants independent of $\mathbf{k} \in Z^d_+$. See [O1] for a different derivation for $d = 2$. Another application is the construction of stable splittings for the modified sparse grid spaces considered in [B].

Remark 2. The above results carry over to analogous constructions for H^1_0 problems. The only difference is that $V_{\mathbf{k},0}$ deteriorates if $k_j = 0$ for some j (which leads to a more complicated notation), and that a modified Franklin system (Schmidt orthogonalization applied to the Faber-Schauder system with the first two functions dropped) has to be used instead.

4. An optimal BPX scheme in the 2D case

To obtain more explicit computational schemes, denote by $\mathcal{N}_{\mathbf{k}} \equiv \{N_{\mathbf{k};i}\}$ the set of nodal basis functions corresponding to $V_{\mathbf{k}}$ for $\mathbf{k} \in Z_+^d$. Denote by $V_{\mathbf{k};i} = \operatorname{span}\{N_{\mathbf{k};i}\}$ the one-dimensional subspaces of $V_{\mathbf{k}}$ corresponding to the nodal basis functions.

The well-known L_2-stability of the nodal basis can be reformulated as

Lemma 4. *If $V_{\mathbf{k}}$ and all $V_{\mathbf{k};i}$ are equipped with the L_2 inner product, scaled by the same factor, then to the splittings (into direct sums of subspaces)*

$$V_{\mathbf{k}} = \sum_i V_{\mathbf{k};i}$$

there correspond two-sided norm estimates (6) with constants c and C that may be chosen independently of $\mathbf{k} \in Z_+^d$.

In other words, we may "refine" a given splitting (involving subspaces $V_{\mathbf{k}}$,equipped with scaled L_2 structures) by using the result of Lemma 4, without destroying the stability constants.

Now, we consider the two-dimensional case in more detail. Lemma 4 and Theorem 1 immediately yield the following splittings that possess stability constants that are uniformly bounded in k :

$$\tilde{V}_k = \sum_{l=0}^{k} V_{k,l}^* = \sum_{l=0}^{[k/2]} V_{l,l} + \sum_{l=[k/2]+1}^{k} (V_{l,k-l} + V_{k-l,l})$$

$$= \sum_{l=0}^{[(k-1)/2]} V_{l,l} + \sum_{l=0}^{k} V_{l,k-l} = \sum_{l=0}^{[(k-1)/2]} \sum_i V_{l,l;i} + \sum_{l=0}^{k} \sum_i V_{l,k-l;i} \quad . \tag{20}$$

Here, the space $\tilde{V}_k$ is equipped with the H^1 inner product while on all other subspaces the L_2 scalar product scaled by the factor 2^{2l} resp. $2^{2\max(l,k-l)}$ is used. Finally, since

$$\|N_{\mathbf{k};i}\|_{H_1}^2 \approx 2^{2|\mathbf{k}|_\infty} \cdot 2^{|\mathbf{k}|_1} \approx 2^{2|\mathbf{k}|_\infty} \cdot \|N_{\mathbf{k};i}\|_{L_2}^2 \tag{21}$$

uniformly in $\mathbf{k}$, and i, we may change back to the H^1 scalar product in the last splitting. Thus, we arrive at

Theorem 2. *Let* (1) *be a symmetric H^1 elliptic variational problem on $\tilde{V}_k$, i.e. a sparse grid discretization of a symmetric second-order elliptic boundary value problem with natural boundary conditions on the unit square of R^2. Then* (1) *can be transformed into an additive Schwarz equation* (10) *where*

$$P_V u = \sum_{l=0}^{[(k-1)/2]} \sum_i \frac{a(u, N_{l,l;i})}{d_{l,l;i}} \cdot N_{l,l;i} + \sum_{l=0}^{k} \sum_i \frac{a(u, N_{l,k-l;i})}{d_{l,k-l;i}} \cdot N_{l,k-l;i} \tag{22}$$

and, in analogy,

$$\phi_V u = \sum_{l=0}^{[(k-1)/2]} \sum_i \frac{\Phi(N_{l,l;i})}{d_{l,l;i}} \cdot N_{l,l;i} + \sum_{l=0}^{k} \sum_i \frac{\Phi(N_{l,k-l;i})}{d_{l,k-l;i}} \cdot N_{l,k-l;i} \tag{22'}$$

with scaling factors $d_{\mathbf{k};i}$ given by

$$d_{\mathbf{k};i} = 2^{2|\mathbf{k}|_\infty} \cdot \|N_{\mathbf{k};i}\|_{L_2}^2 \tag{23}$$

or by

$$d_{\mathbf{k};i} = a(N_{\mathbf{k};i}, N_{\mathbf{k};i}) \quad . \tag{23'}$$

In both cases, the spectral condition number of the Schwarz operator P_V is bounded by a constant which is independent of $k \geq 0$.

With obvious modifications, the results remain valid for H_0^1 variational problems.

We do not go into the details of this exercise (put all norm equivalencies (3), (4), and (6) together, and compute the explicit formulae for the projections $P_{V_{\mathbf{k};i}}u$ etc.). Theorem 2 shows that the above technique of preconditioning the original sparse grid problem (1) by switching to an appropriate Schwarz equation (10), (22)-(22'), (23) (or 23')) is as successful as the corresponding schemes for full grid discretizations (cf. [BPX, Y, Zh]).

To compare this result with our practical experience, we considered the case of the Poisson equation with Dirichlet boundary conditions as a model problem. We computed the eigenvalues $\lambda_{min}(P_V)$, $\lambda_{max}(P_V)$ and the condition number $\kappa(P_V)$ numerically for different values k. We used bilinear basis functions $N_{\mathbf{k};i}$ for the discretization. The results are shown in Table 1.

k	4	5	6	7	8	9	10	11	12	13	14
λ_{min}	0.962	0.862	0.709	0.667	0.621	0.559	0.543	0.535	0.518	0.513	0.510
λ_{max}	2.58	3.71	3.89	4.71	4.92	5.59	5.75	6.28	6.39	6.84	6.93
κ	2.68	4.30	5.48	7.07	7.92	10.0	10.6	11.7	12.3	13.3	13.6

Table 1: Condition numbers of P_V defined by (22) and (23').

From our theoretical results, we know that the condition numbers are bounded by a constant. For practical grid sizes, however, we see still slightly growing condition numbers.

Now, we consider the case of the anisotropic operator $\epsilon^2 \cdot u_{xx} + u_{yy}$. Table 2 shows the resulting eigenvalues and condition numbers of P_V with (22) and (23') for fixed $k = 9$ and varying values of ϵ. We see, that the condition number deteriorates somewhat for $\epsilon \to \infty$, but is still bounded. The same behaviour was obtained for $\epsilon \to 0$. In this sense, our preconditioner is robust. The condition number seems to have an upper bound independent of k and ϵ, at least for a wide range of values of ϵ. In numerical experiments, we found the condition number to deteriorate only near $\epsilon \approx k$. Note furthermore, that the standard BPX preconditioner for the full grid discretization of the anisotropic operator does not possess this robustness property. There, the condition number is independent of k but strongly dependent on ϵ. For example, in the case $k = 6$ and $\epsilon = 1000$, the full grid BPX preconditioner results in a condition number of 2336. For further details, see [GZZ].

ϵ	1	1.4	2	10	100	1000	10000
λ_{min}	0.560	0.549	0.532	0.264	0.503	0.501	0.500
λ_{max}	5.60	6.31	7.14	9.42	11.2	11.5	11.5
κ	10.0	11.5	13.4	35.7	22.3	23.0	23.0

Table 2: Condition numbers of P_V defined by (22) and (23') for anisotropic operator, $k = 9$.

5. A HB scheme for the 2D case

Originally, the sparse grid technique was introduced in a hierarchical basis fashion (see [Z]). To be precise, let

$$I_{\mathbf{k}} = I_{k_1,x_1} \circ \ldots \circ I_{k_d,x_d} : C(\bar{\Omega}) \to V_{\mathbf{k}} \quad , \quad \mathbf{k} \in Z_+^d \, ,$$

denote the usual nodal interpolation projections which can be represented as superposition of one-dimensional interpolation projections onto the corresponding parametrized spline spaces. In other words, $I_{\mathbf{k}}u$ is the unique element of $V_{\mathbf{k}}$ interpolating a continuous function u at the set of nodal grid points $\{P_{\mathbf{k},i}$ associated to the partition $\mathcal{R}_{\mathbf{k}}$. In the present case of linear splines, the order of superposition does not matter. Furthermore, let

$$\Delta I_{\mathbf{k}} = \Delta I_{k_1,x_1} \circ \ldots \circ \Delta I_{k_d,x_d} \qquad , \qquad \mathbf{k} \in Z_+^d \, ,$$

where $\Delta I_{k,x} = I_{k,x} - I_{k-1,x}$, $k > 0$, and $\Delta I_{0,x} = I_{0,x}$. Then any $u \in V = \tilde{V}_k$ possesses a unique decomposition

$$u = \sum_{|\mathbf{m}|_1 \le k} \Delta I_{\mathbf{m}}(u) \equiv \sum_{|\mathbf{m}|_1 \le k} \Delta v_{\mathbf{m}} \tag{24}$$

which will be called sparse hierarchical splitting of u. The image of the projection $\Delta I_{\mathbf{m}}$ will be denoted by $\Delta V_{\mathbf{m}}$. There is a simple basis for $\Delta V_{\mathbf{m}}$ consisting of tensor products of the corresponding hierarchical basis functions (i.e. Faber-Schauder functions) on the interval (see [Z, GSZ, B]). Decomposing the $\Delta v_{\mathbf{m}}$ in (24) further with respect to this basis, we obtain the sparse hierarchical basis representation of $u \in \tilde{V}_k$.

Discretizations with respect to such hierarchical bases are, as a rule, better conditioned than nodal basis discretizations. For full grid discretizations in two dimensions this was shown by Yserentant. Unfortunately, in the sparse grid case, even for the two-dimensional Poisson problem, the condition numbers of the hierarchical discretization grow exponentially. In [L], for the sparse hierarchical basis discretization matrix $\tilde{A}_{HB}$ (which was already diagonally scaled, i.e. $\mathrm{diag}\,(\tilde{A}_{HB}) = I$) the estimates

$$c \cdot 2^{k/2} \le \kappa(\tilde{A}_{HB}) \le C \cdot k^2 \cdot 2^k \quad , \qquad k \to \infty \qquad (d = 2) \tag{25}$$

have been shown. Some improvement of the upper bound is contained in [O5], moreover, it was shown by examples, that the lower bound remains valid for *any* scaling of the hierarchical basis functions. Here, we give a bound which seems to be close to the optimum, and is in agreement with the numerical tests. The proof strongly relies on the technique used in the above Sections, i.e. we use the fact (cf. [Y] for the analogous deduction in the full grid case) that the estimate for $\kappa(\tilde{A}_{HB})$ follows from the estimate for the stability constant of the splitting

$$\tilde{V}_k = \sum_{|\mathbf{m}|_1 \le k} \Delta V_{\mathbf{m}} \qquad , \quad k \ge 0 \tag{26}$$

where $\tilde{V}_k$ is equipped with the H^1 inner product, and $\Delta V_{\mathbf{m}}$ with the L_2 inner product scaled by the factor $2^{2|\mathbf{m}|_\infty}$.

Theorem 3. *For the diagonally scaled sparse hierarchical basis discretization matrix $\tilde{A}_{HB}$ for a symmetric H^1 elliptic variational problem* (1) *on $\tilde{V}_k$ one has*

$$\kappa(\tilde{A}_{HB}) \le C \cdot k^2 \cdot 2^{k/2} \quad , \qquad k \to \infty \qquad (d=2)\ . \tag{27}$$

For H^1_0 problems, an analogous estimate holds true.

Proof. We will prove the equivalent assertion

$$\kappa(\tilde{V}_k, \{\Delta V_{\mathbf{m}} : |\mathbf{m}|_1 \le k\}) \le C \cdot k^2 \cdot 2^{k/2} \quad , \qquad k \to \infty \qquad (d=2) \tag{27'}$$

for the stability constant of the splitting (26) by relying on the results of Theorem 1 resp. 2 for $d = 2$. Let $u \in \tilde{V}_k$ be given and consider its hierarchical splitting (24):

$$u = \sum_{|\mathbf{m}|_1 \le k} \Delta v_{\mathbf{m}} = \sum_{l=0}^{k} \overbrace{\sum_{|\mathbf{m}|_1 \le k, |\mathbf{m}|_\infty = l} \Delta v_{\mathbf{m}}}^{u_l^*}$$

Since $u_l^* \in V_{k,l}^*$, we can use Theorem 1 :

$$\begin{aligned} \|u\|^2_{H^1} &\le 1/c^* \sum_{l=0}^{k} 2^{2l} \cdot \|u_l^*\|^2_{L_2} \\ &\le 1/c^* \sum_{l=0}^{k} 2^{2l} \cdot \hat{c} \cdot k \sum_{|\mathbf{m}|_1 \le k, |\mathbf{m}|_\infty = l} \|\Delta v_{\mathbf{m}}\|^2_{L_2} = 1/c^* \cdot \hat{c} \cdot k \sum_{|\mathbf{m}|_1 \le k} 2^{2|\mathbf{m}|_\infty} \cdot \|\Delta v_{\mathbf{m}}\|^2_{L_2} \end{aligned}$$

(from here on, c^*, C^* denote the constants in the norm equivalence (6) corresponding to the splittings of Theorem 1 resp. 2 while $\hat{c}$ is a generic constant). This is the desired lower estimate in (6) for the hierarchical splitting (26).

For the upper estimate, let

$$u = \sum_{l=0}^{[(k-1)/2]} u_{l,l} + \sum_{l=0}^{k} u_{l,k-l} \equiv \sum_{\mathbf{m} \in \tilde{\Gamma}} u_{\mathbf{m}}$$

be an arbitrary decomposition with respect to the splitting (20). Note that

$$\tilde{\Gamma} = \{(l,l) \,:\, l \le [(k-1)/2]\} \cup \{(l, k-l) \,:\, l = 0, \dots, k\}$$

has $k+1+[(k+1)/2] \le 2k+1$ elements. Since, by definition of the interpolation projections and the hierarchical splitting,

$$\Delta v_{\mathbf{k}} = \Delta I_{\mathbf{k}}(u) = \sum_{\mathbf{k-1} \le \mathbf{r} \le \mathbf{k}} \pm I_{\mathbf{r}}(u_{\mathbf{k},\tilde{\Gamma}}) \qquad \text{with} \quad u_{\mathbf{k},\tilde{\Gamma}} = \sum_{\mathbf{m} \in \tilde{\Gamma}, \mathbf{k} \le \mathbf{m}} u_{\mathbf{m}}\ ,$$

we have

$$\|\Delta v_{\mathbf{k}}\|_{L_2}^2 \leq 4 \cdot \sum_{\mathbf{k}-\mathbf{1}\leq \mathbf{r}\leq \mathbf{k}} \|I_{\mathbf{r}}(u_{\mathbf{k},\tilde{\Gamma}})\|_{L_2}^2 \,.$$

But by the L_2 stability of the nodal bases in $V_{\mathbf{m}}$ (cf. Lemma 4)

$$\begin{aligned}
\|I_{\mathbf{r}}(u_{\mathbf{k},\tilde{\Gamma}})\|_{L_2}^2 &\leq \hat{c}\cdot 2^{-|\mathbf{r}|_1}\cdot \sum_i |u_{\mathbf{k},\tilde{\Gamma}}(P_{\mathbf{r};i})|^2 \\
&\leq \hat{c}\cdot k\cdot 2^{-|\mathbf{r}|_1}\cdot \sum_{\mathbf{m}\in\tilde{\Gamma},\mathbf{k}\leq\mathbf{m}} \sum_i |u_{\mathbf{m}}(P_{\mathbf{r};i})|^2 \\
&\leq \hat{c}\cdot k\cdot 2^{-|\mathbf{k}|_1}\cdot \sum_{\mathbf{m}\in\tilde{\Gamma},\mathbf{k}\leq\mathbf{m}} 2^{|\mathbf{m}|_1}\|u_{\mathbf{m}}\|_{L_2}^2 \qquad \text{for} \quad \mathbf{k}-\mathbf{1}\leq\mathbf{r}\leq\mathbf{k}\,.
\end{aligned}$$

After substituting we can complete the reasoning :

$$\begin{aligned}
\sum_{|\mathbf{k}|_1\leq k} 2^{2|\mathbf{k}|_\infty}\cdot\|\Delta v_{\mathbf{k}}\|_{L_2}^2 &\leq \hat{c}\cdot k \sum_{|\mathbf{k}|_1\leq k} 2^{2|\mathbf{k}|_\infty-|\mathbf{k}|_1}\cdot \sum_{\mathbf{m}\in\tilde{\Gamma},\mathbf{k}\leq\mathbf{m}} 2^{|\mathbf{m}|_1}\|u_{\mathbf{m}}\|_{L_2}^2 \\
&\leq \hat{c}\cdot k\cdot \sum_{\mathbf{m}\in\tilde{\Gamma}} \left\{ 2^{|\mathbf{m}|_1}\|u_{\mathbf{m}}\|_{L_2}^2 \cdot \sum_{\mathbf{k}\leq\mathbf{m}} 2^{2|\mathbf{k}|_\infty-|\mathbf{k}|_1} \right\} \\
&\leq \hat{c}\cdot k\cdot \sum_{\mathbf{m}\in\tilde{\Gamma}} 2^{|\mathbf{m}|_1+|\mathbf{m}|_\infty}\|u_{\mathbf{m}}\|_{L_2}^2 \leq \hat{c}\cdot k\cdot 2^{k/2}\cdot \sum_{\mathbf{m}\in\tilde{\Gamma}} 2^{2|\mathbf{m}|_\infty}\|u_{\mathbf{m}}\|_{L_2}^2 \,.
\end{aligned}$$

It remains to take the infimum with respect to all allowed decompositions, and to refer to Theorem 2. This gives the upper bound

$$\sum_{|\mathbf{k}|_1\leq k} 2^{2|\mathbf{k}|_\infty}\cdot\|\Delta v_{\mathbf{k}}\|_{L_2}^2 \leq C^*\cdot\hat{c}\cdot k\cdot 2^{k/2}\cdot\|u\|_{H^1}^2 \,,$$

and together with the lower bound we have (27') as well as the assertion of Theorem 3.

Remark 4. The proof of Theorem 3 shows that

$$c\cdot k^{-1}\leq \lambda_{\min}(\tilde{A}_{HB})\leq\lambda_{\max}(\tilde{A}_{HB})\leq C\cdot k\cdot 2^{k/2} \qquad , \quad k\to\infty \quad (d=2).$$

To compare this bound with our practical experiences, once more, we considered the case of the Poisson equation with Dirichlet boundary conditions as a model problem. We computed the eigenvalues of the hierarchical basis system matrix in the unscaled (A_{HB}) and scaled case ($\tilde{A}_{HB}$) for different values k numerically . The results are shown in Table 3.

	k	4	5	6	7	8	9	10	11
	λ_{min}	0.736	0.595	0.490	0.400	0.314	0.258	0.215	0.181
A_{HB}	λ_{max}	6.1	10.9	21.4	42.7	85.4	170.6	341.3	682.7
	κ	8.3	18.3	43.7	106.9	271.5	661.2	1587	3771
	λ_{min}	0.249	0.186	0.135	0.098	0.0665	0.0484	0.0331	0.0241
$\tilde{A}_{HB}$	λ_{max}	1.77	2.12	2.55	2.90	3.34	3.69	4.12	4.47
	κ	7.1	11.4	18.9	29.5	50.2	76.2	124.5	185.5

Table 3: Eigenvalues and condition numbers for A_{HB} and $\tilde{A}_{HB}$.

We clearly see the improvement achieved by diagonal scaling. The condition number of $\tilde{A}_{HB}$ behaves like $k\cdot 2^{k/2}$ which is a factor k better than the bound given in Theorem

3. Thus, we guess that in the upper bound (for $\lambda_{\max}(\tilde{A}_{HB})$) the factor k can be removed. However, this is not important for practical applications, anyway. The exponential growth, expressed by the factor $2^{k/2} \approx h_k^{-1/2}$, shows the superiority of sparse BPX over sparse HB in this respect.

6. Concluding remarks

For the higher dimensional case, the proofs of the statements in Section 4 and 5 may be repeated almost line by line. In the BPX case we arrive at the splitting

$$\tilde{V}_k = \sum_{\mathbf{m}=(m,\ldots,m)\,:\,dm<k} V_{\mathbf{m}} + \sum_{\mathbf{m}\,:\,|\mathbf{m}|_1=k} V_{\mathbf{m}} \equiv \sum_{\mathbf{m}\in\tilde{\Gamma}_d} V_{\mathbf{m}} \quad , \quad \text{with} \quad (\cdot,\cdot)_{V_{\mathbf{m}}} = 2^{2|\mathbf{m}|_\infty} \cdot (\cdot,\cdot)_{L_2}$$

for which one obtains the stability estimate

$$\kappa(\tilde{V}_k, \{V_{\mathbf{m}} \,:\, \mathbf{m} \in \tilde{\Gamma}_d\}) \leq C \cdot k^{d-2} \quad , \qquad k \to \infty \, . \tag{28}$$

Thus, in the 3D case our theoretical approach leads to one additional logarithmic factor which appears in the last but one step of the decomposition procedure described in (20). We do not know whether this factor can be removed by some other splitting that is better suited for the 3D case.

For the sake of completeness, note that our methods also imply condition number estimates of the corresponding diagonally scaled 3D sparse hierarchical basis discretization which deteriorate to a $O(k^5 \cdot 2^{2k/3})$ bound. Surprisingly, this is asymptotically better than in the 3D full grid situation.

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