Adaptive Hierarchical Methods for Landscape Representation and Analysis

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Abstract. Hierarchical interpolation techniques allow the efficient representation of digital terrain data. Due to the inherent adaptivity of these methods, less points are needed for the storage of smooth areas compared to non–adaptive methods. They also allow the derivation of approximate terrain models with variable level of detail. This results in a significant compression of the data as well as a highly reduced computational cost of algorithms for terrain analysis.
We will present two out of a large variety of methods, both using regular structured grids. The hierarchical triangulation method uses a successively refinable triangular grid based on right triangles. The sparse grid method, on the other hand, uses a tensor–product approach and has a better theoretical complexity. Based on these two approaches, we also construct multivariate biorthogonal wavelets using the lifting scheme. We compare all methods by means of of numerical examples.

1 Introduction

Digital elevation models (DEMs) are the starting point for many geographic applications like geographic information systems, topographic analysis, climatic, hydrological or geomorphological simulations, and landscape visualization. Most of these models are based on sampled elevation data on a rectangular grid (e.g. obtained by satellite images, radar or laser altimetry, or topographic maps) together with an interpolation method which allows the derivation of elevation values in between the grid points.

1.1 Multiresolution Models

Especially, if the grid has a small mesh size and covers a large area, the amount of data to be stored and processed can easily surmount the capabilities of today's computers. Compression algorithms try to lower or minimize storage requirements by removing redundancy in the data which is usually very prominent in DEMs. For example, in plain areas like lakes the elevation does not change, however the same elevation value is stored many times.

If the compression is lossless, the original model can be reconstructed exactly from the stored data. Otherwise, only an approximate DEM can be derived. The approximative approach usually allows much better compression rates and can be quite satisfying as long as there is some kind of control on the error caused by compression.

If the error is uniformly distributed over the whole domain, the reconstructed DEM is a representation of the original DEM at a coarser scale. In many applications such as multilevel computations it is necessary to access the data at a variety of scales. Schemes allowing several representations of the original data at various resolutions are called multiresolution models.

In this paper, we consider algorithms for the construction of such multiresolution models and for the reconstruction of approximate terrain models from these models with arbitrary accuracy. The resolution can even vary over the whole domain, allowing coarse representations in less interesting areas, finer approximations in more interesting areas and smooth transitions between the different levels of detail.

We present two methods based on adaptive hierarchical interpolation, a hierarchical triangulation method and the so–called sparse grid approach. Both methods use regular structured grids implying that the location of the grid points and their neighbour relations are fixed. Multiresolution models based on hierarchically structured grids allow fast construction and reconstruction by simple recursive algorithms. They also easily allow the parallel processing and distributed storage of the data, but their application is restricted to rectangular domains. Unstructured grid methods on the other hand are more flexible but computationally more expensive and require more memory per grid point stored.

The outline of this paper is as follows. In section 2 we will consider the hierarchical interpolation of functions in one dimension and compare the hierarchical basis with the standard nodal basis. Generalizations to two or more dimensions are presented in section 3. Here, we study the hierarchical triangulation and the sparse grid approaches in detail. We compare these methods with the help of some numerical examples. Section 4 generalizes these approaches towards biorthogonal wavelets, which can be used for scale and frequency analysis of terrains. We apply all four methods to the compression of a DEM located at the Eifel mountains. In section 5, we conclude with some remarks on further generalizations of the methods presented.

1.2 Related Work

The issue of deriving more compact representations for DEMs is a special topic of the more general problem of polygonal simplification of three–dimensional objects. It is being addressed in a variety of fields, such as computer graphics, image processing, computer aided design, computational geometry, approximation theory, and numerical mathematics.

The most popular approaches for terrain simplification are unstructured triangle meshes called TINs (triangular irregular networks), as introduced by (Fowler and Little 1979). The algorithms for the construction of TINs can be roughly classified as decimation and refinement methods. In the first approach, starting from the data at full resolution, points are removed and/or relocated until the desired accuracy is reached. The latter approach starts with a simple coarse model and inserts points successively until the approximation error is less than the prescribed error threshold. For a recent survey on simplification methods see (Heckbert and Garland 1997).

Multiresolution terrain models can be classified as hierarchical and pyramidal models. Hierarchical models are based on a hierarchical decomposition of the domain by recursive partitions, such as the methods we propose. Pyramidal models on the other hand are based on several approximate DEMs with different levels of detail, which are e.g. constructed by a refinement or decimation algorithm. A survey paper on multiresolution models was compiled by (Puppo and Scopigno 1997).

2 Hierarchical Interpolation of Functions

The idea of hierarchical interpolation of functions is very old. In fact, it goes at least back to Archimedes who computed π approximately using a hierarchy of polygonal approximations of a circle. At the beginning of our century this approach has again been studied in approximation theory by (Faber 1909). But it has not been brought to a broader attention until the eighties as an important ingredient for the efficient solution of partial differential equations (Zienkiewicz, Gago and Kelly 1983, Yserentant 1986).

The main idea behind hierarchical interpolation is to start with a coarse overall representation of a given function and then to improve this approximation by adding local corrections at varying scales until a certain error criterion is satisfied. We will first consider the one–dimensional case in this section, and then look at possible extensions to higher dimensions in the next section.

2.1 Problem Formulation

We assume that for a given (height) function f the input data set contains $N := 2^L + 1$ values $f(x_j), 0 \le j \le N$, on an equidistant grid $\{x_j\}$ on $\Omega = [0, 1]$ with mesh width $h_L := 2^{-L}$. For simplicity of the presentation and without loss of generality we assume that $f(x_0) = f(x_N) = 0$. The interpolation problem is then to find a function $\tilde{f}$ interpolating f at the points x_j, i.e.

$$\tilde{f}(x_j) = f(x_j).$$

Usually, $\tilde{f}$ is constructed by a linear combination of (e.g. Lagrange, Hermite, spline, ...) basis functions $\{\phi_j\}$,

$$\tilde{f}(x) := \sum_{j=1}^{N-1} c_j \cdot \phi_j(x).$$

Once the basis is chosen, the coefficients c_j can be found by solving the corresponding linear system of equations

$$\Phi \, \mathbf{c} = \mathbf{f},$$

where $\Phi := (\phi_k(x_j))_{jk}$, $\mathbf{c} := (c_j)_j$ and $\mathbf{f} := (f(x_j))_j$.

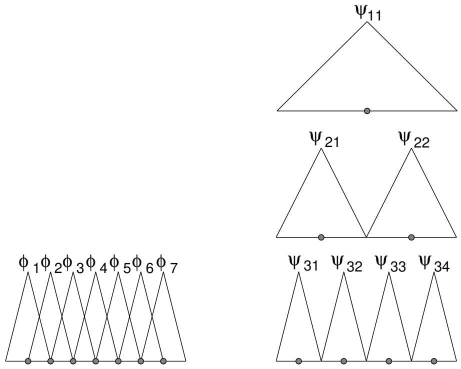

Fig. 1. Piecewise linear nodal and hierarchical basis for $L = 3$

2.2 Piecewise Linear Nodal Basis

The coefficients c_j are particularly easy to compute if a nodal (Lagrange) basis is used, i.e.

$$\phi_k(x_j) = \delta_{jk} = \begin{cases} 1 \text{ for } j = k \\ 0 \text{ for } j \neq k \end{cases}.$$

The coefficients c_j are then simply given by

$$c_j = f(x_j).$$

A simple choice for the $\{\phi_j\}$ are linear spline basis functions (fig. 1, left) which are given by

$$\phi_j(x - x_j) := \max\left\{1 - 2^{L+1}|x - x_j|,\ 0\right\}.$$

This leads to a piecewise linear interpolant $\tilde{f}$ on the grid $\{x_j\}$.

2.3 Piecewise Linear Hierarchical Basis

The nodal basis construction does not take the smoothness of f into account. In order to accomplish this, we reorder the grid points in a certain hierarchical fashion, i.e. we assign to each grid point a level $l, 1 \leq l \leq L$. Then, we label the grid points with an index pair $(l, i), 1 \leq i \leq 2^{l-1}$, and define a bijective index transformation $\tau(l, i)$ by

$$\tau(l, i) := 2^{L-l} \cdot (2i - 1).$$

The same transformation can be applied to the basis functions, now labeled ψ_{li} (fig.1, right). The interpolant is then constructed by

$$\tilde{f}(x) := \sum_{l=1}^{L} \sum_{i=1}^{2^{l-1}} u_{li} \cdot \psi_{li}(x) = \sum_{j=1}^{N-1} u_{\tau^{-1}(j)} \psi_{\tau^{-1}(j)}(x).$$

In order to make these coefficients efficiently computable, we use a nodal basis condition for points on the same level and we demand that the basis functions vanish on lower level points,

$$\psi_{li}(x_{kj}) = \begin{cases} \delta_{ij}, \text{ if } & k = l \\ 0, \quad \text{if } & k < l. \end{cases}$$

This leads to a linear system of equations with a lower triangle matrix (if the x_{kj} are ordered with increasing k) which can be easily inverted. If we again choose a piecewise linear basis, the hierarchical linear spline basis functions $\{\psi_{li}\}$ (fig. 1, right) are given by

$$\psi_{li}(x - x_{li}) := \max\left\{1 - 2^l|x - x_{li}|,\ 0\right\}.$$

In this case the coefficients u_{li} can even be computed in constant time and can be expressed in stencil notation by

$$u_{li} = \left[-\frac{1}{2} \ \ 1 \ \ -\frac{1}{2}\right]_{x_{li},h_l} f \;= -\frac{1}{2}f(x_{li} - h_l) + f(x_{li}) - \frac{1}{2}f(x_{li} + h_l).$$

The coefficients u_{li} can be considered as the difference between the value of f at the grid point x_{li} and the linear interpolant of the values of f from neighbouring grid points. Therefore they are also called "hierarchical surplus". Since all basis functions are piecewise linear on the grid $\{x_j\}$, this approach also leads to a piecewise linear interpolant.

2.4 Comparison of the nodal and hierarchical basis

Since our interpolation problem has a unique solution, both approaches result in the same interpolant. We calculate the interpolation errors in the L_∞–norm for functions g defined by

$$||g||_\infty := \sup_{x \in \Omega} |g(x)|.$$

If f has the smoothness property that $f \in C^2(\Omega)$ and the L_∞–seminorm of f defined by

$$|f|_{2,\infty} := \left\| \frac{\partial^2 f}{\partial x^2} \right\|_\infty$$

is bounded, then, for the L_∞–norm of the interpolation error,

$$||f - \tilde{f}||_\infty = O(h_L^2),$$

holds. In this case, in contrast to the coefficients c_j, the coefficients u_{li} decrease with the level l, more exactly,

$$|u_{li}| = O(4^{-l}).$$

Considering functions f which do not have the smoothness property over the whole domain, the coefficients u_{li} nevertheless get relatively small, where f is

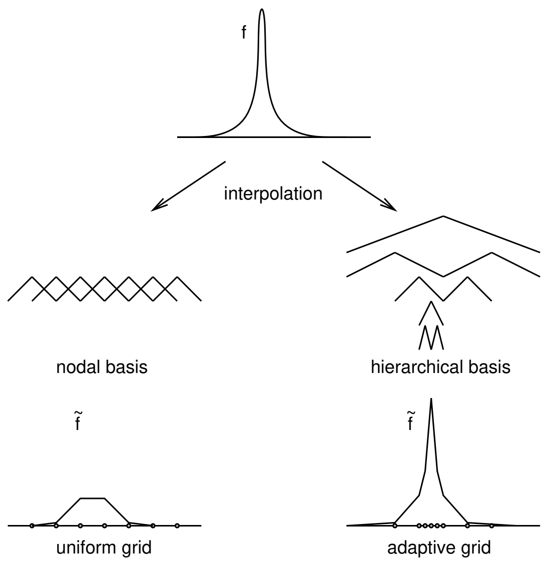

Fig. 2. Interpolation of a peak function using a nodal basis and adaptively selected hierarchical basis functions

locally smooth and are relatively large, where f is not smooth. This property can be exploited by adaptive algorithms (see fig. 2). These algorithms spend more points in areas where f is not smooth to reach a prescribed error bound with less points than non–adaptive grids do. In contrast to the nodal basis construction, the number of grid points (and therefore basis functions) can grow locally during the construction process of the interpolant.

Note that instead of the hierarchical piecewise linear basis, a hierarchical piecewise constant basis can be used, which is useful e.g. for image compression. One can also construct hierarchical polynomial bases of higher order and derive better results for the interpolation error for function classes having a higher degree of smoothness (Forsey and Bartels 1988, Bungartz 1996).

This can be used as an alternative form of adaptivity. Instead of the mesh width, the polynomial degree of the bases is considered to be variable depending on the local smoothness of f. Spatial and polynomial adaptivity can even be combined resulting in error estimates having exponential instead of polynomial decay (Babuska and Suri 1994).

2.5 Compression and Approximation

The adaptivity of the hierarchical basis can be used for data compression and approximation. For this, we first transform the whole data set into a hierarchical representation by computing the coefficients u_{li}. Then, points whose absolute coefficients $|u_{li}|$ are smaller than a prescribed cutoff value ε_{cut} are removed and only the remaining coefficients are stored. The reconstruction step takes the

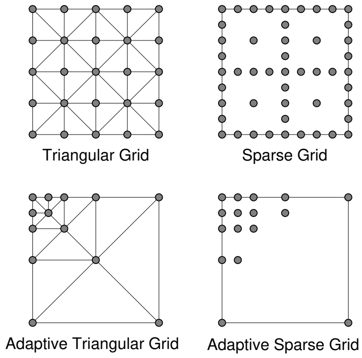

Fig. 3. Examples for regular and adaptive two–dimensional grids

stored coefficients and computes the values $\tilde{f}(x_{li})$ if the corresponding coefficient u_{li} has been stored. If it has not been stored, the value $\tilde{f}(x_{li})$ is interpolated linearly from the already computed neighbouring values.

The reconstructed data is an approximation of the original data with an approximation error depending on the cutoff value ε_{cut}. The approximation error is globally not strictly bounded by ε_{cut}, because local errors can accumulate in the hierarchical representation if no smoothness is assumed for the data. In practice, however, error accumulation is quite low.

The most efficient algorithms use a bottom–up approach for data compression. They start with the points on the maximum level $l = L$ and compute the hierarchical coefficients u_{li} for each level until the lowest level $l = 1$. The reconstruction is then performed in a top–down fashion, starting with the lowest level to the maximum level. Both, compression and reconstruction can be achieved in $O(N)$ operations and can thus be applied in real time for fairly large data sets.

3 Generalizations to 2–D

There are many possibilities to generalize the one–dimensional hierarchical basis approach to two or more dimensions. For reasons of simplicity, we assume that our domain Ω is the unit square $[0,1]^2$ with f being 0 on the boundary $\partial\Omega$. This is no restriction, because it is possible to apply the univariate method on the boundaries. The interpolant is again constructed as a linear combination of basis functions. The basis functions and interpolation stencils we will consider are suitable generalizations of their one–dimensional counterparts. We pick up two examples which both have interesting properties.

The first approach is based on a hierarchical triangulation of the domain. The corresponding grids are called triangular grids. The basis functions are pyramidal hat functions, leading to a piecewise linear interpolant. The second approach uses a tensor–product subspace decomposition of the unit square which leads to so–called sparse grids. The basis functions are bilinear hat functions, leading to a piecewise bilinear interpolant (fig. 3 and 4).

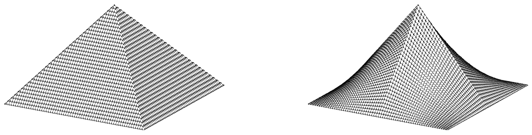

Fig. 4. Pyramidal and bilinear hat function

3.1 Hierarchical Triangulation

The presented hierarchical triangulation approach is commonly called "recursive bisection method" and goes back to (Rivara 1984) and (Mitchell 1991) who used it for the solution of elliptic partial differential equations. It is based on right triangles and is closely related to restricted quadtrees. It has been applied to terrain simplification and visualization by (Gerstner 1995), (Lindstrom et al. 1996), (Paul and Dobler 1997) and (Evans, Kirkpatrick and Townsend 1997).

It starts with an initial triangulation of the unit square and then cuts recursively each triangle by the longest side. This approach also defines a hierarchy on the grid points being the vertices of the triangulation (fig. 5). In this figure we have also given the grid points on the border of the domain to reveal the underlying structure of the construction despite the condition of f being zero on $\partial\Omega$. We set $\mathbf{x} = (x, y)$ and identify the grid points by their level l and a two–dimensional index $\mathbf{i}$. Furthermore, we set $\mathbf{x}_{l\mathbf{i}} =: (u, v)$. If a piecewise linear approach is taken, the basis functions are pyramidal hat functions (fig. 4), defined for even l by

$$\psi_{l\mathbf{i}}(\mathbf{x} - \mathbf{x}_{l\mathbf{i}}) = \max\{\min\{1 - 2^{l/2}|x - u|, 1 - 2^{l/2}|y - v|\}, 0\}$$

and for odd l by a version rotated by 90 degrees,

$$\psi_{l\mathbf{i}}(\mathbf{x} - \mathbf{x}_{l\mathbf{i}}) = \max\{\min\{1 - 2^{(l-1)/2}|x - u + y - v|, 1 - 2^{(l-1)/2}|x - u - y + v|\}, 0\}.$$

The hierarchical surplus can be computed by interpolating each new grid point linearly along the side of the triangle it is placed on using the one–dimensional 3–point–stencil, eventually rotated by 90, 180, or 270 degrees (fig. 6).

The approximation properties of this approach are well-known, e.g. from finite element analysis (Braess 1992). If $f \in C^2(\Omega)$ and all partial derivatives of f of order 2 are bounded,

$$\left\| \frac{\partial^2 f}{\partial x^i \partial y^j} \right\|_\infty < \infty, \quad \text{for} \quad i, j \geq 0 \quad \text{with} \quad i + j = 2,$$

then, for the approximation error

$$\|f - \tilde{f}\|_\infty = O(h_L^2)$$

holds. Here, the number of grid points is of order $O(h_L^{-2})$.

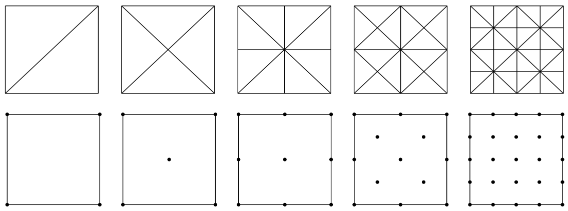

Fig. 5. Triangulation hierarchy (levels 1 to 5) and corresponding grid points

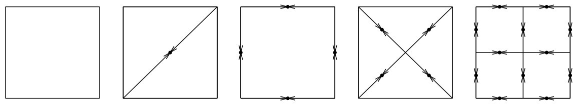

Fig. 6. Vertex dependencies in the hierarchical triangulation

Since we have now constructed a hierarchical basis, adaptivity also is possible. But in this construction, the underlying adaptive triangulation requires a certain regularity property. It is necessary that the levels of two adjacent triangles must not differ by more than 1 (which is e.g. true in fig. 3). If this is not the case, our representation for f using continuous basis functions is not valid and discontinuities in the approximation can arise.

3.2 Sparse Grids

The sparse grid approach goes back to (Smolyak 1963), who used it for numerical integration of functions. Since then it has been applied to numerous fields such as computer graphics (Gordon 1969), interpolation (Delvos and Schempp 1989), and the solution of partial differential equations (Zenger 1991, Griebel 1991).

The main construction principle for sparse grids is a tensor product approach. Starting from our one–dimensional hierarchical basis, all possible products of basis functions are considered and combined into a two–dimensional tableau (see fig. 7). To describe the sparse grid construction formally, we replace the single indices from the one–dimensional case by index pairs. The two–dimensional basis functions are then bilinear hat functions (fig. 4) with varying supports in x- and y-direction, defined by

$$\psi_{\mathbf{li}}(\mathbf{x}) = \psi_{l_1 i_1}(x) \cdot \psi_{l_2 i_2}(y).$$

The hierarchical surplus can be computed using the analogously defined tensor product of the one–dimensional stencils,

$$u_{\mathbf{li}} = \begin{bmatrix} -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}_{x_{l_1 i_1}, h_{l_1}} \otimes \begin{bmatrix} -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}_{x_{l_2 i_2}, h_{l_2}} f = \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} \end{bmatrix}_{x_{\mathbf{li}}, h_{\mathbf{l}}} f.$$

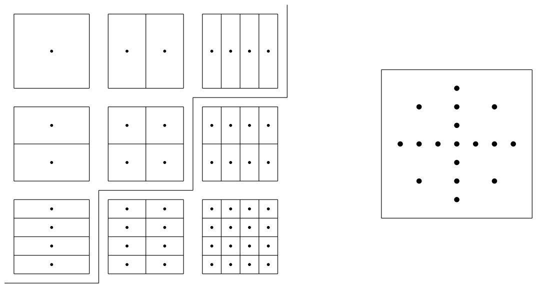

Fig. 7. Supports of the tensor–product basis functions and sparse grid

Alternatively, the hierarchical surpluses can be computed in a two step process. In the first step, one–dimensional surpluses are computed for each row of the original data. Then, the one–dimensional stencil is applied to the just computed one–dimensional surpluses column–wise yielding the two–dimensional surpluses. Of course, the role of columns and rows can be exchanged. This procedure is computationally cheaper than using the 9–point stencil if data access is fast.

The error analysis reveals some interesting properties of this construction. Instead of the intuitive square scheme, where all basis functions with $l_1, l_2 \leq L$ are selected, a triangular scheme can be used by selecting only the basis functions above the diagonal $l_1 = l_2$ (fig. 7). The interpolant is therefore constructed by

$$\tilde{f}(\mathbf{x}) := \sum_{l_1+l_2 \leq L+1} \sum_{i_1=1}^{2^{(l_1-1)}} \sum_{i_2=1}^{2^{(l_2-1)}} u_{\mathbf{li}} \cdot \psi_{\mathbf{li}}(\mathbf{x}).$$

The centers of the domains of the selected basis functions form a sparse grid (fig. 7). In this case, the approximation error grows slightly compared to the full square scheme, but the number of basis functions is substantially reduced (Zenger 1991, Bungartz 1992). More exactly, if the second partial mixed derivative of f is bounded,

$$\left\| \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial y^2} f \right\|_\infty < \infty,$$

then, for the approximation

$$\|f - \tilde{f}\|_\infty = O(h_L^2 \log h_L^{-1})$$

holds with the number of grid points only of order $O(h_L^{-1} \log h_L^{-1})$.

Since we have again formed a hierarchical basis, we can easily construct adaptive sparse grids (fig. 3). However, unlike in the hierarchical triangulation construction, discontinuities in the interpolant can not arise.

	Hier. Triang.	Hier. Triang. Wav.	Sparse Grids	Sparse Grid Wav.
30 m	0.02 %	0.02 %	0.03 %	0.03 %
20 m	0.21 %	0.26 %	0.23 %	0.25 %
10 m	1.10 %	1.13 %	0.91 %	1.02 %
5 m	3.59 %	3.70 %	3.17 %	3.17 %
2 m	9.94 %	10.41 %	10.87 %	9.87 %
1 m	22.11 %	25.59 %	30.04 %	29.11 %
0 m	55.85 %	51.43 %	72.57 %	51.43 %

Table 1. Relative number of grid points used for the Eifel DEM and various accuracies with the four presented methods

3.3 Comparison

Both approaches can be generalized easily to arbitrary dimensions. The sparse grid approach has a better theoretical complexity, but the computation of the hierarchical surpluses is more expensive. The hierarchical triangulation method usually performs better in less smooth areas and allows the efficient graphical representation of the interpolant because of the greater locality of the basis functions.

3.4 Numerical Examples

We tested our methods on a DEM with 257×257 grid points and grid resolution 50×50 meters located in northwestern Germany at the northern border of the Eifel mountains. The maximum elevation difference is about 400 meters (fig. 8). We computed approximative DEMs with varying accuracies between 0 and 30 meters using the adaptive hierarchical methods of the previous sections.

For the 10 and 5 meters accuracies, the sparse grid method performs slightly better than the hierarchical triangulation method. In the case of the higher accuracy DEMs, the sparse grid method requires more points than the hierarchical triangulation method (tab. 1). The required grid points for the 10, 5, and 2 meter accuracies are depicted for both approaches in (fig. 10 and 11).

In all cases, the construction of the multiresolution model and the reconstruction of the approximate DEM needed less than 0.1 seconds on a SGI O2 R10000, 150 MHz. A perspective view of a part of the area using about 15000 triangles is shown in (fig. 9). In this picture, the error has not been uniformly distributed over the whole domain but has been weighted according to the distance to the viewpoint. This implies a greater level of detail near the observer and a lower level of detail further away.

In landscape visualization many approximate models with varying accuracies have to computed. In this case, it is possible to speed up computations by computing and storing additional information in each grid point and restricting the traversal of the data set to the visible parts (Paul 1997).

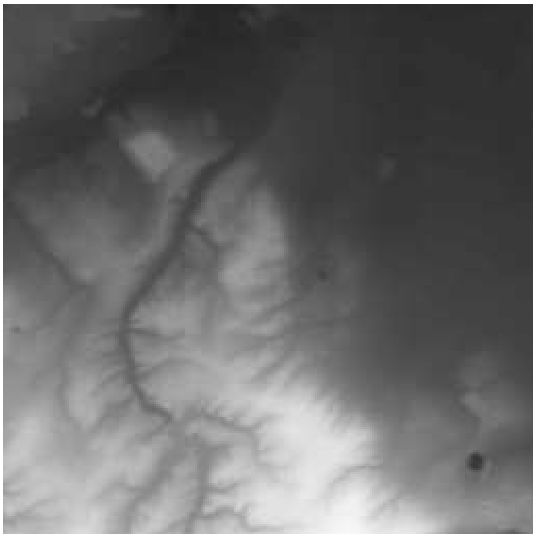

Fig. 8. Hypsoshaded DEM at the northern border of the Eifel mountains

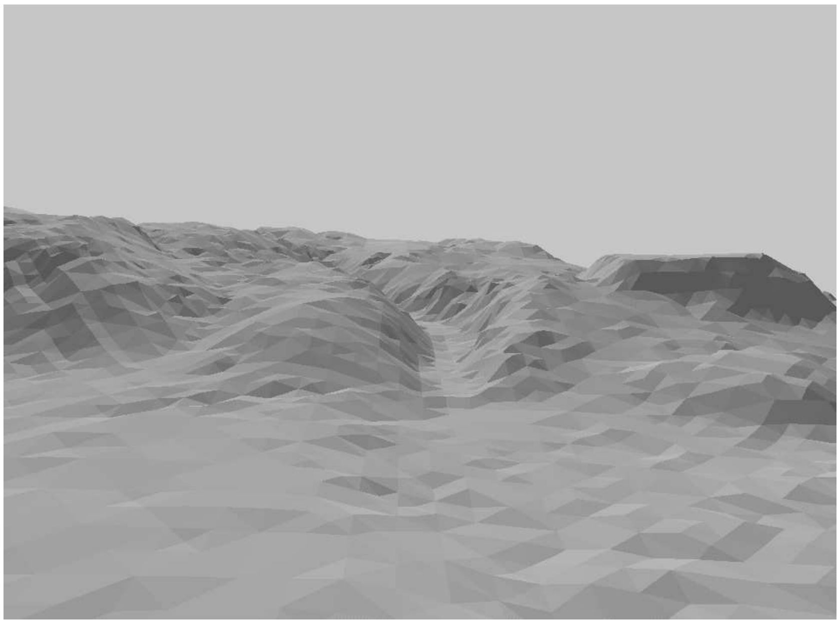

Fig. 9. Perspective view of the DEM looking from the upper border towards the river using flat shading

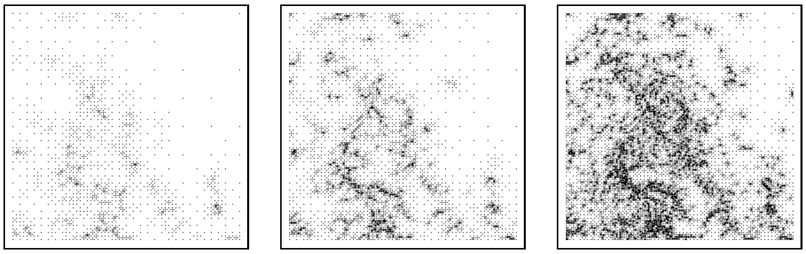

Fig. 10. Adaptive triangular grids with accuracy 10 m ($\sim 1.1\%$), 5 m ($\sim 3.6\%$) and 2 m ($\sim 9.9\%$ of the original grid points) for the Eifel DEM

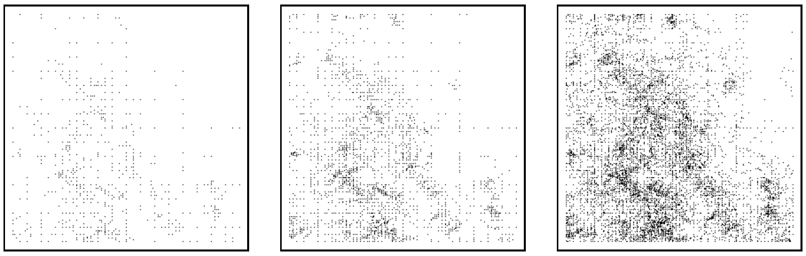

Fig. 11. Adaptive sparse grids with accuracy 10 m ($\sim 0.9\%$), 5 m ($\sim 3.2\%$) and 2 m ($\sim 10.9\%$ of the original grid points) for the Eifel DEM

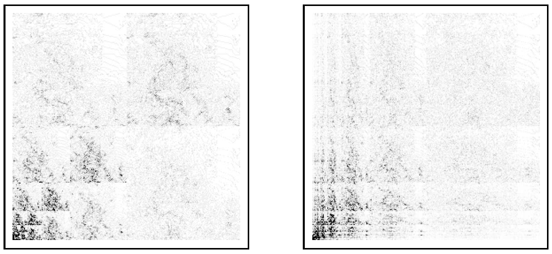

Fig. 12. Magnitude of wavelet coefficients computed by the hierarchical triangulation and the sparse grid method with lifting for the Eifel DEM

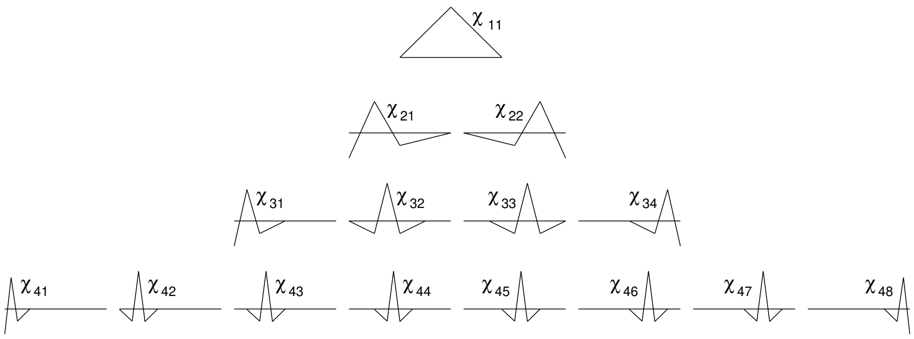

Fig. 13. Biorthogonal wavelet basis constructed by lifting for $L = 4$

4 Landscape Analysis using Wavelets

The hierarchical coefficients allow some notion about the local smoothness of a terrain. On the other hand, one is also often interested in information about local frequencies of the data. This is not possible with the help of the Fourier transformation because of its missing spatial locality. The wavelet transformation is an efficiently computable localized transformation into frequency domain (Daubechies 1992, Chui 1992). The application of wavelets and multiresolution analysis to terrain and surface simplification is described e.g. by (Gross, Staadt and Gatti 1996) and (Eck et al. 1995).

With our hierarchical bases we have already taken the first step towards wavelets. Now we have to modify the construction to achieve such properties like orthogonality or vanishing moments. A univariate function g has m vanishing moments, if

$$\int g(x) \cdot x^k dx = 0$$

for $0 \leq k \leq m$. We will use a construction called "lifting" (Sweldens 1996) with piecewise linear bases, which allows us to construct biorthogonal wavelets with vanishing first moments (Cohen, Daubechies and Feauveau 1992).

The construction starts with the given input data on an equidistant grid and uses a bottom–up approach. Starting with the highest level for each level it first computes the hierarchical surplus for the grid points on the current level. But then it corrects ("lifts") the values on lower levels. This is performed in such a way that the integral of the corresponding wavelet basis functions is zero. Since the basis functions are symmetric in the interior of the domain, they will also have vanishing first moments.

We again start with the one–dimensional case. After the hierarchical ordering of the values $f_{li} := f(x_{li})$, and setting $u(x_{li}) := u_{li}$ for each level $l = L$ down to 1 the following operations are executed:

$$\begin{aligned} u_{li} &:= \begin{bmatrix} -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}_{x_{li},h_l} f \quad \text{for} \quad 1 \leq i \leq 2^{l-1} \\ f_{ki} &:= \begin{bmatrix} \frac{1}{4} & 1 & \frac{1}{4} \end{bmatrix}_{x_{ki},h_l} u \quad \text{for} \quad 1 \leq k < l,\ 1 \leq i \leq 2^{k-1} \end{aligned}$$

Values close to the boundary of the interval require a modification of the lifting stencil. Since one of the values u_{li} is not present for the computation of the f_{ki}, the corresponding coefficient is set to zero while the other coefficient is set to $\frac{1}{2}$. This results in the basis functions $\chi_{11} = \psi_{11}$ and for $l > 1$

$$\chi_{li}(x) := -\frac{1}{4}\psi_{\tau^{-1}(l,i)-2^{L-l}}(x) + \psi_{li}(x) - \frac{1}{4}\psi_{\tau^{-1}(l,i)+2^{L-l}}(x)$$

in the interior of the interval and with the coefficients $\frac{1}{4}$ being changed to $\frac{1}{2}$ at the boundary of the interval (fig. 13). Note that the indices $\tau^{-1}(l,i) \pm 2^{L-l}$ refer to the neighbouring points of x_{li} on the level l.

The reconstruction is simply done top–down from level $l = 1$ to L by inverting the order of the operations and changing the sign of all fractions.

For the multivariate case, the same constructions of the previous chapter can be applied. In the sparse grid approach, the lifting stencil is also the tensor product of the one–dimensional stencil. For the hierarchical triangulation the lifting stencil has to be rotated just like the hierarchical surplus stencil, and the coefficients in the stencil are $\frac{1}{8}$ in the interior of the domain, $\frac{1}{4}$ at the edges, and $\frac{1}{2}$ at the corners. Other tensor product and triangular wavelet constructions can be found in (Griebel and Oswald 1995) and (Kotyczka and Oswald 1995).

In (fig. 12) we show the wavelet coefficients for the Eifel DEM computed by the hierarchical triangulation and sparse grid methods with lifting. The wavelet coefficients are reordered hierarchically starting with the lowest level coefficients in the lower left corner to the highest level coefficients in the upper right corner of the pictures. We see that in both cases the wavelet coefficients mainly decrease with the level. High frequency peaks like the one in the lower right corner of the domain are detected more clearly by the hierarchical triangulation method. The sparse grid method, on the other hand, reveals the vertical and horizontal structure of the DEM much better.

The wavelet transformation can also be used for data compression (see section 2.5) by considering wavelet coefficients instead of hierarchical coefficients. The computing time is about twice as large as without lifting, but the compression results are just about the same (tab. 1) for the DEM tested.

5 Further Extensions

There are several possible extensions to the methods presented. Instead of biorthogonal wavelets, one can choose semi–orthogonal or orthogonal wavelets, which have a better location in frequency domain, but are more costly to implement. There are also many other biorthogonal wavelet constructions some of which can be constructed using lifting by applying other lifting stencils, including higher order schemes which allow more vanishing moments.

In some applications it might be desirable, that certain regions of interest are approximated better than other regions, independently of the local smoothness. This can be achieved by making the cutoff value dependent on the location of the grid point considered, that is $\varepsilon_{cut} = \varepsilon_{cut}(x)$. This corresponds to a local weighting of the underlying basis functions (see also fig. 8).

The topology of the terrain can be preserved better by using different error criteria, e.g. the difference of the normals of two adjacent triangles.

Very large data sets which do not fit into main memory can be treated by a coarse block–structuring of the data set and by loading only the currently needed blocks into memory.

So far, we have only considered the elevation data for compression. But if attributes of the terrain, like humidity or temperature are given as an attribute function over the domain Ω, they can be treated analogously. By comparing their wavelet coefficients, even correlations between attributes and the elevation data can be derived. This information can be valuable for the analysis of the data and can be used for further data compression by considering the whole data set and coding the correlations appropriately.

It is possible to construct more flexible methods by relaxing some of our initial restrictions. Without the interpolation condition $\tilde{f}(x_i) = f(x_i)$ further approximative methods can be derived by minimizing the resulting error functional. It is also possible to use a redundant basis (often called "frame" (Daubechies 1992) or "generating system" (Griebel 1994)) or even so–called "best–basis" techniques (Wickerhäuser 1996) to represent f and select basis functions from this system.

All presented methods can be generalized to arbitrary dimensions. Time–dependent data (e.g. dynamical changes of the terrain or of the humidity of the terrain) can be represented by treating time as a third dimension, if there is a finite number of time steps. For the sparse grid case, the savings compared to the full grid are then even more dramatic than in the two–dimensional case, if all mixed derivatives of order less than $2d$ are bounded. More exactly, by denoting d as the dimension, only $O(N \log N^{d-1})$ points are needed compared to $O(N^d)$ for almost the same accuracy.

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