EXTRAPOLATION, COMBINATION, AND SPARSE GRID TECHNIQUES FOR ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract. Several variants of extrapolation can be used for elliptic partial differential equations. They are *Richardson extrapolation*, *truncation error extrapolation* and *extrapolation of the functional*. In multi-dimensional problems, multivariate error expansions can be exploited by *multivariate extrapolation*, where the asymptotic expansions in different mesh parameters are exploited. Particularly interesting cases are the *combination technique* that uses all the grids that have constant product of the meshspacings in the different coordinate directions. Another related technique is the *sparse grid finite element technique* that can be interpreted as a combination extrapolation of the functional.

Key words. Elliptic partial differential equations, asymptotic expansions, extrapolation, sparse grids, combination technique.

1. Introduction.

Extrapolation and related techniques. High order methods for elliptic boundary value problems may be based on extrapolation and related techniques. This topic has been studied in the report Rüde [14].

Extrapolation is an attractive device for obtaining high order, because it is based on a general principle that is independent of the particular context. Ideally, a simple low order discretization, combined with extrapolation, delivers the same approximation quality as involved high order methods. As only the basic discretization is problem dependent, the algorithm is easily adaptable to new situations. Extrapolation for elliptic equations can be naturally combined with multilevel and multigrid techniques.

Extrapolation can be applied to elliptic equations in three conceptually different versions. First, we have its classical form *Richardson extrapolation*, see Marchuk and Shaidurov [8], and Blum, Lin, and Rannacher [2], where linear combinations of approximate solutions for different mesh sizes are formed. Besides this *explicit* scheme, there are two *implicit* extrapolation schemes. One is based on the *extrapolation of the truncation error*, where asymptotic expansions of the truncation error are exploited, see Rüde [13]. The other scheme is based on the *extrapolation of the functional*. Both methods result in an implicit definition of the solution and are usually implemented by defect correction, see Blum [1]. For the extrapolation of the functional, we use the variational formulation and the observation that the discretization can be interpreted as the process of constructing numerical approximations to the functional. The error in these approximations can be expressed by asymptotic series. When extrapolation can

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be applied to the functional, new discrete equations are defined explicitly. This can be analyzed based on local properties and can be shown to be equivalent to higher order finite elements, see Rüde [13, 16, 15].

For problems in two or more dimensions, the mesh parameters for the different coordinate directions can be chosen separately. This leads to *multivariate* extrapolation schemes, where, depending on the respective approach, the coefficients of the multivariate asymptotic series of the error, the truncation error, or the functional approximation are eliminated by combining suitable approximate solutions for different mesh sizes, see Lin and Schüller [17] and Rüde [14].

An approach closely related to multivariate extrapolation leads to reduced approximation spaces, where no longer conventional *full grids* with $O(h^{-d})$ degrees of freedom, but so-called *sparse grids* with only $O(h^{-1}|\log h|^{d-1})$ degrees of freedom are used. Here, d denotes the number of dimensions of the problem. These sparse grid spaces have been introduced in Zenger [19] and were analyzed further in Bungartz [3]. In the sparse grid context, explicit multivariate extrapolation leads to the so-called *combination technique*, see Griebel, Schneider, and Zenger [6], and a special case of implicit multivariate extrapolation results in the *sparse grid finite element technique*, see Zenger [19] and Bungartz [3]. The combination technique and the sparse grid finite element method are characterized by $O(h^2|\log h|^{d-1})$ errors (in L_2) and $O(h)$ errors (in H^1) obtained with only $O(h^{-1}|\log h|^{d-1})$ degrees of freedom. Except for the logarithmic factors, these methods are equivalent in complexity to fourth order methods. With additional extrapolation, still higher order can be obtained for both approaches.

Because of its relation to finite elements, extrapolation of the functional can be analyzed within the theoretical framework of finite element theory. In particular, it does *not* depend on the existence of global expansions, but can be applied in the same generality as the p- and hp-versions of the finite element method. The extrapolation of the functional avoids the high setup and solution cost of the p- and hp-versions of the finite element method but can be shown to have equivalent approximation properties. In particular, it is applicable on nonuniform meshes and is suitable for locally refined meshes, leading to h- and p-adaptive methods.

Notation and basic results. For reasons of simplicity, we pick a simple model problem. We study finite element techniques for the solution of the homogeneous Dirichlet problem of Poisson's equation

$$\begin{aligned} -\Delta u &= f(x,y) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \\ u &\in H_0^1(\Omega), \end{aligned} \tag{1}$$

where we assume that $\Omega = (0,1)^2 \subset \mathbb{R}^2$ is the unit square. Equation (1) is discretized by finite elements. We will focus on *rectangular elements* of length $h_x,\ h_x = 1/n_x,\ n_x \in \mathbb{N}$, and width $h_y,\ h_y = 1/n_y,\ n_y \in \mathbb{N}$, and on the corresponding uniform grids G^{h_x,h_y},

$$G^{h_x,h_y} := \{(ih_x, jh_y),\ \ 1 \le i \le n_x - 1,\ 1 \le j \le n_y - 1\}.$$

Using piecewise bilinear trial functions, we finally get the spaces $S_0^{h_x,h_y} \subset H_0^1(\Omega)$ of all continuous functions that are bilinear in each element.

Now, a numerical approximation to equation (1) is obtained by replacing the continuous space $H_0^1(\Omega)$ by one of its approximations $S_0^{h_x,h_y}$. This leads to a symmetric, positive definite linear system

$$A^{h_x,h_y} u^{h_x,h_y} = f^{h_x,h_y}, \tag{2}$$

where A^{h_x,h_y} is the so-called *stiffness matrix* for the described type of discretization. This matrix has a nine-banded structure.

Asymptotic expansions. Under suitable assumptions of regularity, it can be shown that the solution u of an elliptic problem allows a *multivariate asymptotic expansion* of the form

$$u^{h_x,h_y} = u + h_x^2 e^{2,0} + h_y^2 e^{0,2} + h_x^4 e^{4,0} + h_x^2 h_y^2 e^{2,2} + \cdots \tag{3}$$

with expressions $e^{2,0}$, $e^{0,2}, \ldots$ that depend on x and y, but that do neither depend on h_x nor on h_y, see Volkov [18], Hofmann [7], Rüde [11], and Röschke [9].

2. Explicit extrapolation.

2.1. Richardson extrapolation. We now exploit asymptotic expansions for the approximate solutions. In the simplest case, we use expansion (3) only for $h = h_x = h_y$, where it pointwise reduces to

$$u^{h,h} = u + h^2 e_1 + h^4 e_2 + \cdots,$$

and we define the *extrapolated* solutions by

$$\begin{aligned} u_0^{h,h} &:= u^{h,h}, \\ u_k^{h,h} &:= \frac{2^{2k}}{2^{2k}-1} I_{h/2,h/2}^{h,h} u_{k-1}^{h/2,h/2} + \frac{1}{1-2^{2k}} u_{k-1}^{h,h}, \end{aligned}$$

where $I_{k_x,k_y}^{h_x,h_y} : S_0^{k_x,k_y} \to S_0^{h_x,h_y}$ is the injection operator defined by

$$(I_{k_x,k_y}^{h_x,h_y} u^{k_x,k_y})(x,y) = u^{k_x,k_y}(x,y) \ \text{ for all } (x,y) \in G^{h_x,h_y}. \tag{4}$$

Here, the order of accuracy for $u_k^{h,h}$ is $O(h^{2k+2})$. However, note that Richardson expansion is expensive, because for each step of extrapolation an additional problem with four times the size of the previous one must be solved.

2.2. Multivariate Richardson extrapolation. In order to reduce the cost of extrapolation, we re-examine the expansions with separate h_x and h_y and study linear combinations of solutions of the form

$$u_{\text{extra}}^{h_x,h_y} := \sum_{i=0}^{p} \sum_{j=0}^{q} \alpha_{ij} I_{2^{-i}h_x,2^{-j}h_y}^{h_x,h_y} u^{2^{-i}h_x,2^{-j}h_y}$$

with constant coefficients α_{ij}. The goal of this linear combination is to eliminate as many terms in the asymptotic expansion as possible spending as little work as possible.

At a first glance, the advantage of having more terms available to perform extrapolation is compensated by having more terms in the expansion, because terms in h_x and h_y appear separately. For example, to eliminate the h_x^2-term and the h_y^2-term, we may use a combination like

$$u_{\text{extra}}^{h_x,h_y} = \frac{4}{3} \cdot I_{h_x/2,h_y}^{h_x,h_y} u^{h_x/2,h_y} + \frac{4}{3} \cdot I_{h_x,h_y/2}^{h_x,h_y} u^{h_x,h_y/2} - \frac{5}{3} \cdot u^{h_x,h_y}.$$

This extrapolation scheme has been suggested by Schüller and Lin [17], where it is called *splitting extrapolation*. The order of accuracy for $u_{\text{extra}}^{h_x,h_y}$ is now $O(h_x^2 \cdot h_y^2 + h_x^4 + h_y^4)$. If $h_x = h_y = h$, this is $O(h^4)$, just as for the usual extrapolation involving the two terms $u^{h/2,h/2}$ and $u^{h,h}$. The modified extrapolation technique needs three terms involving solutions of problems with N, $2N$, and $2N$ unknowns, whereas classical Richardson extrapolation involves solutions of a problem with N and one with $4N$ unknowns.

The multivariate extrapolation will have a complexity advantage, if we study higher dimensional problems or if we wish to extrapolate further. We can order the error terms in a matrix like

$$(5) \qquad \begin{array}{cccccc} 1 & h_x^2 & h_x^4 & h_x^6 & h_x^8 & \cdots \\ h_y^2 & h_x^2 h_y^2 & h_x^4 h_y^2 & h_x^6 h_y^2 & \cdots & \\ h_y^4 & h_x^2 h_y^4 & h_x^4 h_y^4 & \cdots & & \\ h_y^6 & h_x^2 h_y^6 & \cdots & & & \\ h_y^8 & \cdots & & & & \end{array}$$

and the approximate solutions in a corresponding table like

$$(6) \qquad \begin{array}{ccccc} u^{h_x,h_y} & u^{h_x/2,h_y} & u^{h_x/4,h_y} & u^{h_x/8,h_y} & \cdots \\ u^{h_x,h_y/2} & u^{h_x/2,h_y/2} & u^{h_x/4,h_y/2} & \cdots & \\ u^{h_x,h_y/4} & u^{h_x/2,h_y/4} & \cdots & & \\ u^{h_x,h_y/8} & \cdots & & & \end{array}.$$

If we wish to eliminate the error terms $h_x^{2i} h_y^{2j}$ for $2i + 2j \leq 2p$, as indicated by the triangular shape of the scheme in (5), we can use a combination of solutions $u^{h_x 2^{-i}, h_y 2^{-j}}$ for $0 \leq i$, $0 \leq j$ and $i+j \leq p$, as indicated in (6). Note that this choice of $(p+2)(p+1)/2$ approximate solutions to be calculated is the cheapest possible in order to eliminate the indicated error terms.

Assume u^{h_x,h_y} is computed from a system with N unknowns, such that $u^{h_x 2^{-i}, h_y 2^{-j}}$ involves $2^{i+j} N$ unknowns. For comparing the efficiency of the multivariate extrapolation with the regular one we assume optimal solvers, whose work estimate is proportional to the number of unknowns. The total work estimate is therefore given by summing the unknowns of all problems to be solved. We have

$$(7) \qquad W = \sum_{i=0}^{p} \sum_{j=0}^{p-i} 2^{i+j} N = N(2^{p+1} p + 1) = N \cdot O(p 2^p)$$

for the multivariate extrapolation to order h^{2p+2} ($h_x = h_y = h$), as compared to

$$W = \sum_{i=0}^{p} 4^i N = N\left(\frac{4^{p+1}}{3} - \frac{1}{3}\right) = N \cdot O(4^p)$$

for the classical extrapolation to order h^{2p+2} involving $u^{h2^{-i},h2^{-i}}$ for $i = 0, \ldots, p$. Thus, for increasing p multivariate extrapolation will become much cheaper. This is particularly true in a parallel environment, because multivariate extrapolation can be easily parallelized, see Griebel, Huber, Rüde, and Störtkuhl [5].

The extrapolation coefficients for complicated schemes can be generated by symbolic computer algebra programs, see Rüde [14].

2.3. The combination technique. In Griebel, Schneider, and Zenger [6] another type of extrapolation, the so-called *combination technique* is introduced. It is not based on asymptotic expansions like in (3), but only requires the existence of an *error splitting* of the form

$$u^{h_x,h_y} - u = e_x(h_x) + e_y(h_y) + R(h_x, h_y), \tag{8}$$

where e_x just depends on h_x, x, and y, whereas e_y just depends on h_y, x, and y. Furthermore,

$$|R(h_x, h_y)| \leq c \cdot (h_x h_y)^\nu \tag{9}$$

with constants c and ν independent of h_x, h_y, x, and y. For sufficiently smooth functions u, we get $\nu = 2$ and

$$e_x(h_x) = O(h_x^2), \qquad e_y(h_y) = O(h_y^2).$$

With these assumptions we can study combinations of the form

$$\hat{u}_k^{h_x,h_y} := \sum_{i=1}^{k} u^{h_x 2^{-i}, h_y 2^{i-k-1}} - \sum_{i=1}^{k-1} u^{h_x 2^{-i}, h_y 2^{i-k}}. \tag{10}$$

Note that the combined solution $\hat{u}_k^{h_x,h_y}$ is not defined on a regular full grid, but on a so-called *sparse grid* (see section 3). In scheme (6), (10) corresponds to taking the sum of the solutions along a diagonal with $h_x \cdot h_y = H$ and subtracting the solutions of the neighboring diagonal with $h_x \cdot h_y = 2H$. Substituting the error splitting (8) into the combination formula (10) and re-ordering the terms gives

$$\hat{u}_k^{h_x,h_y} - u = e_x(2^{-k} h_x) + e_y(2^{-k} h_y) + \hat{R},$$

where

$$\begin{aligned} |\hat{R}| &= \left| \sum_{i=1}^{k} R(2^{-i} h_x, 2^{i-k-1} h_y) - \sum_{i=1}^{k-1} R(2^{-i} h_x, 2^{i-k} h_y) \right| \\ &\leq \hat{c} \cdot \left(k 2^{-(k+1)\nu} + (k-1) 2^{-k\nu}\right) (h_x h_y)^\nu. \end{aligned} \tag{11}$$

If we use the same assumptions to derive error estimates for an equivalent *full grid* $G^{2^{-k}h_x, 2^{-k}h_y}$, we get

$$u^{2^{-k}h_x, 2^{-k}h_y} - u = e_x(2^{-k}h_x) + e_y(2^{-k}h_y) + \tilde{R},$$

where

$$|\tilde{R}| \leq \tilde{c} \cdot (2^{-k}h_x 2^{-k}h_y)^{\nu}.$$

Note that the combined solution has the same error terms e_x and e_y as the solution on grid $G^{2^{-k}h_x, 2^{-k}h_y}$. If these are the dominating terms, then the combined solution will have an accuracy comparable to that of the full grid solution. In the fully regular case, we have $e_x(h_x) = O(h_x^2)$, $e_y(h_y) = O(h_y^2)$, and $\nu = 2$, so that $\hat{R}$ in equation (11) will be of order $O(k2^{-2k}h_x^2 h_y^2)$.

Assuming the complexity of the solver is proportional to the number of unknowns, the work estimate for finding $u_k^{1,1}$ is (see also equation (7))

$$(12) \qquad \begin{aligned} W_k &= \sum_{j=1}^{k}(2^j - 1)(2^{k+1-j} - 1) + \sum_{j=1}^{k-1}(2^j - 1)(2^{k-j} - 1) \\ &= 3k \cdot 2^k - 7 \cdot 2^k + 2k + 7 \;=\; O(k2^k) \end{aligned}$$

for $k >= 2$.

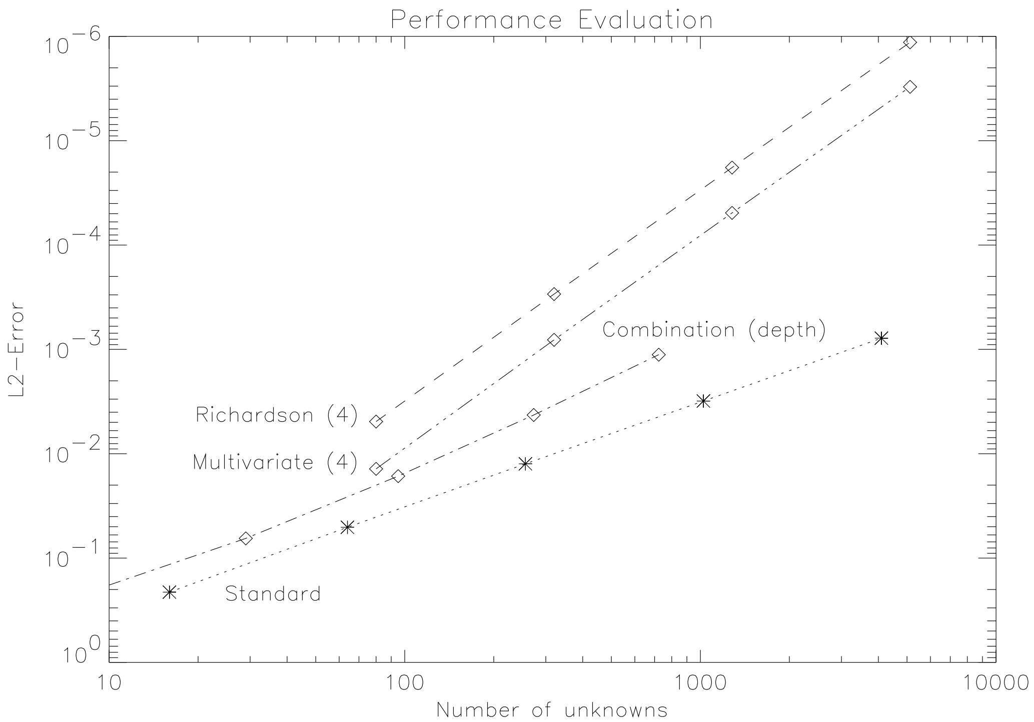

FIG. 1. *Performance of Richardson-like extrapolation methods.*

Comparison of explicit extrapolation methods. In Figure 1, we display the performance of various explicit extrapolation methods by plotting the accuracy vs. the number of unknowns. The graph clearly shows the superiority of higher order methods. The slope of the lines reflects the order, so that the methods of 4th order have parallel lines. The combination solution produces a curved line, sloping upward, reflecting the fact that for comparatively small numbers of unknowns the logarithmic factors in complexity and accuracy are still essential. For a larger number of unknows, the combination method should asymptotically provide an order close to 4 (with respect to the number of unknowns). Therefore, the graph should asymptotically tend to the same slope as the 4th order methods.

Note that the combination idea can be applied to higher order solutions, if those satisfy an error splitting as in equation (9) with accordingly large ν.

3. Sparse grids. In this section, we describe a method that is closely related to the combination technique. We will introduce the so-called *sparse grid finite element method* by applying the Ritz-Galerkin formalism to the spaces defined by the combination technique.

The set of unknowns for the combination technique is described by the *combined grid*

$$G_k^{h_x,h_y} := \bigcup_{i=1}^{k} G^{2^{-i}h_x,2^{i-k-1}h_y}. \tag{13}$$

The special case $G_k^{1,1}$ is the *sparse grid* introduced by Zenger [19]. As an example, Figure 2 shows the sparse grid $G_7^{1,1}$. A sparse grid corresponds to a union of finite

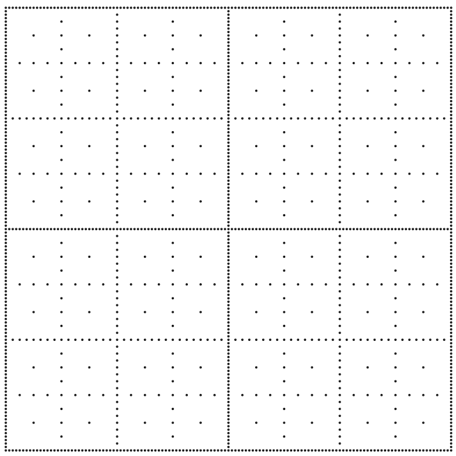

FIG. 2. *Sparse grid $G_7^{1,1}$ (including points on the boundary)*

element spaces

$$S_{0,k}^{1,1} := \bigcup_{i=1}^{k} S_0^{2^{-i},2^{i-k-1}}.$$

In Bungartz [3], the properties of these sparse grid spaces $S_{0,k}^{1,1}$ are analyzed. Especially, a sufficiently smooth function (i.e., a function u where $\frac{\partial^4 u}{\partial x^2 \partial y^2}$ exists and is continuous) can be represented on $G_k^{1,1}$ with an error of the order $O(k \cdot 2^k)$ (in L_2) and $O(2^k)$ (in H_0^1). In addition to these interpolation qualities and in contrast to the combination technique, an error analysis based on finite element approximation theory is possible for sparse grids (see Bungartz [3]), giving estimates of the form

$$\|u_k^{1,1} - u\| < c 2^{-k},$$

where u denotes the analytical solution, $u_k^{1,1} \in S_{0,k}^{1,1}$ denotes the sparse grid approximate solution, c is a constant independent of k, and $\|\cdot\|$ is the energy norm. We see that the error estimate is $O(h)$, where $h = 2^{-k}$ is the smallest width of the finite elements used in the sparse grid space $S_{0,k}^{1,1}$. As an error estimate of $O(h)$ is also obtained in the full grid space $S_0^{h,h}$, the finest conventional space containing $S_{0,k}^{1,1}$, the quality does not deteriorate significantly when we use sparse grids.

Note that in numerical experiments, the L_2-error of the approximate sparse grid solution $u_k^{1,1}$ shows the $O(h^2|\log h|)$-behaviour we could expect from the interpolation properties of sparse grids mentioned above (see Table 1, "S"). Furthermore, by two simple modifications of the standard sparse grid finite element method, L_2-errors of a still higher order can be achieved. Profiting from the hierarchical representation of functions in the sparse grid context and simulating infinitely fine grids, the first variant gives an L_2-accuracy of $O(h^2)$. A second improvement results in $O(h^4|\log h|)$-errors, using higher order trial functions. Note that, with respect to the reduced number of unknowns, $O(h^4)$ on sparse grids corresponds to $O(h^8)$ on full grids. Table 1 shows some numerical results for a smooth model problem. For details, see Bungartz [3].

TABLE 1
L_2-errors on $G_k^{1,1}$: Standard (S), variant I (I), variant II (II)

k=	7	8	9	10	11	12
S	$2.506 \cdot 10^{-5}$	$6.401 \cdot 10^{-6}$	$1.631 \cdot 10^{-6}$	$4.147 \cdot 10^{-7}$	$1.053 \cdot 10^{-7}$	$2.673 \cdot 10^{-8}$
I	$2.834 \cdot 10^{-6}$	$6.821 \cdot 10^{-7}$	$1.643 \cdot 10^{-7}$	$3.971 \cdot 10^{-8}$	$9.641 \cdot 10^{-9}$	$2.350 \cdot 10^{-9}$
II	$2.520 \cdot 10^{-8}$	$1.793 \cdot 10^{-9}$	$1.231 \cdot 10^{-10}$	$8.360 \cdot 10^{-12}$	$5.619 \cdot 10^{-13}$	$3.769 \cdot 10^{-14}$

Though the structure of the stiffness matrix for a sparse grid space $S_{0,k}^{1,1}$ is more complicated than for a conventional discretization, see Figure 3, we note that optimal solvers (see Griebel [4]) lead to a total work proportional to the number of unknowns given by (see Bungartz [3] and Rüde [14])

$$W_k = 1 + (k-1) \cdot 2^k.$$

Asymptotically, the sparse grid system only involves a third of the unknowns necessary in the combination technique, see (12). Whether this can compensate the additional complexity introduced by the more complicated structure of the linear system, is an open problem. Possibly, the much more important advantage of the sparse grid technique is its extensibility to adaptive refinement, as studied by Bungartz [3].

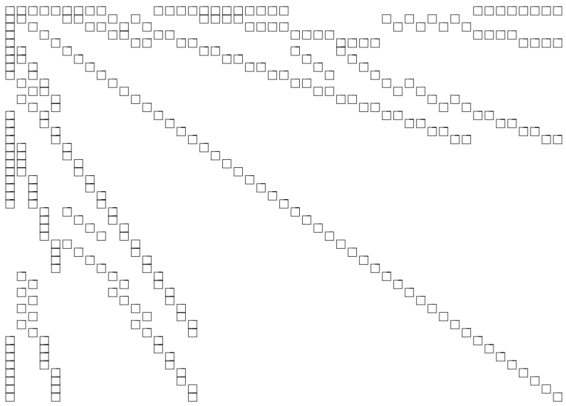

FIG. 3. *Sparsity pattern for sparse grid stiffness matrix*

4. Energy Extrapolation. The finite element solution to a given problem can be interpreted as an approximate solution of a minimization problem. Instead of the classical interpretation that the discrete system (2) is obtained by restricting the space of admissible functions $H_0^1(\Omega)$ to a subspace $S_0^{h_x,h_y}$, we can also interpret the discrete system as an *approximation to the true functional* $E(u)$ on the full space, obtained by replacing any function u by the vector of its nodal values $\vec{u}$ and inserting it into the discrete quadratic form

$$E^{h_x,h_y}(u) := \frac{1}{2}\vec{u}^{\,T} A^{h_x,h_y}\vec{u} - \vec{f}^{\,T}\vec{u}. \tag{14}$$

Higher order methods should be obtained by attempting to provide more accurate approximations to the true functional $E(u)$.

In fact, under suitable assumptions of regularity for the function u and the coefficients of the equation, and for rather general finite element situations (in particular without assuming uniform meshes), we can show (see Rüde [12]) that the *discrete energy* (14) has expansions that are suitable for extrapolation.

Exploiting the expansions we may study linear combinations of discrete functionals of the form

$$\bar{E}_J(u) := \sum_{(h_x,h_y)\in J} \alpha_{h_x,h_y} E^{h_x,h_y}(u), \tag{15}$$

where J denotes the set of functionals participating in the extrapolation, and where the coefficients α_{h_x,h_y} are chosen such that the extrapolated functional has higher order accuracy.

Energy extrapolation in a nested system of spaces. In the most straightforward case, when an h^2-Richardson extrapolation for the energy is used and

$$S^{2^k h_x, 2^k h_y} \subset S^{2^{k-1} h_x, 2^{k-1} h_y} \subset \ldots \subset S^{h_x,h_y},$$

the linear systems become

$$A_k^{h_x,h_y} u_k^{h_x,h_y} = f_k^{h_x,h_y}, \tag{16}$$

where the matrices $A_k^{h_x,h_y}$ are defined recursively by

$$A_0^{h_x,h_y} := A^{h_x,h_y}, \tag{17}$$

$$A_k^{h_x,h_y} := \frac{2^{2k}}{2^{2k}-1} A_{k-1}^{h_x,h_y} + \frac{1}{1-2^{2k}} (I_{h_x,h_y}^{2h_x,2h_y})^T A_{k-1}^{2h_x,2h_y} (I_{h_x,h_y}^{2h_x,2h_y}). \tag{18}$$

Here, $I_{k_x,k_y}^{h_x,h_y} : S_0^{k_x,k_y} \rightarrow S_0^{h_x,h_y}$ is the injection operator defined by (4). Again, a function $u \in S_0^{h_x,h_y}$ is identified with the vector of its nodal values. The right hand sides $f_k^{h_x,h_y}$ must be defined analogously. Table 2 presents the results (discrete L_2-norms) of a numerical experiment for the solution of the energy-extrapolated systems (16). It can be seen clearly that one step of extrapolation (u_1) improves the accuracy to $O(h^4)$, but a second step (u_2) fails to improve the order. This is caused by highly oscillatory solution modes whose accuracy is not represented correctly by extrapolation. Each step of extrapolation increases the space of correctly represented polynomials by one degree. As the number of unknowns grows exponentionally, we will get spurious degrees of freedom causing the oscillations. A general remedy will be studied further below. We begin by discussing a model case.

TABLE 2
Energy extrapolation for a smooth model problem

h	u_0	u_1	u_2	$\bar{u}$
1/8	4.739e-2	1.190e-2	-	5.138e-3
1/16	1.164e-2	8.266e-4	1.374e-3	3.231e-4
1/32	2.899e-3	5.382e-5	9.364e-5	2.021e-5
1/64	7.241e-4	3.425e-6	6.063e-6	1.255e-6

Quadratic functions are characterized by a basis with 6 elements

$$1, x, y, x^2, xy, y^2. \tag{19}$$

The grid generated by extrapolation to represent these basis functions correctly consists of *macro elements* with 3×3 nodes, corresponding to the basis functions in (19) *plus*

$$x^2y, xy^2, x^2y^2. \tag{20}$$

They can be additionally represented correctly on a nine-node macro element, if we use a multivariate extrapolation based on the four grids

$$G^{h_x,h_y}, G^{2h_x,h_y}, G^{h_x,2h_y}, G^{2h_x,2h_y}.$$

With $H_x = 2h_x$ and $H_y = 2h_y$, the corresponding system is

$$\begin{aligned} &\Big(\frac{16}{9}A^{h_x,h_y} - \frac{4}{9}\Big((I_{h_x,h_y}^{h_x,H_y})^T A^{h_x,H_y}(I_{h_x,h_y}^{h_x,H_y}) + (I_{h_x,h_y}^{H_x,h_y})^T A^{H_x,h_y}(I_{h_x,h_y}^{H_x,h_y})\Big) \\ (21) \qquad &\qquad +\frac{1}{9}(I_{h_x,h_y}^{H_x,H_y})^T A^{H_x,H_y}(I_{h_x,h_y}^{H_x,H_y})\Big)\,\bar{u}^{h_x,h_y} \quad = \\ &\Big(\frac{16}{9}f^{h_x,h_y} - \frac{4}{9}\Big((I_{h_x,h_y}^{h_x,H_y})^T f^{h_x,H_y} + (I_{h_x,h_y}^{H_x,h_y})^T f^{H_x,h_y}\Big) + \frac{1}{9}(I_{h_x,h_y}^{H_x,H_y})^T f^{H_x,H_y}\Big). \end{aligned}$$

This extrapolation formula eliminates all the error terms h_x^2, h_y^2, and $h_x^2 h_y^2$. The corresponding solution $\bar{u}^{h_x,h_y}$ is equivalent to a finite element solution with the basis functions (19) and (20). Numerical results are reported in column $\bar{u}$ of Table 2.

If we wish to extend the energy extrapolation to higher accuracy, we must solve the problem of the oscillatory modes which result from the dimension of the discrete space growing faster than the space of improved accuracy. We can remove the spurious modes whose accuracy is not improved by extrapolation if we subject the functional minimization to constraints that have a smoothing (that is stabilizing) effect, i.e. by restricting the space of test functions so that the spurious modes are excluded (see Rüde [10, 14]). Instead of directly implementing the constraint, we can add a penalty term with a suitable weighting factor ϵ to the functional (15) for each $\tilde{u} \in \tilde{S}$:

$$\bar{E}_J(\tilde{u}) + \epsilon||D\tilde{u}||^2.$$

We set up an extrapolation based on 16 terms

$$A_J^{1/64,1/64} = \sum_{k=1}^{4}\sum_{l=1}^{4} \alpha_{k,l} I_{2^k/64,2^l/64}^{1/64,1/64} A^{2^k/64,2^l/64} I_{1/64,1/64}^{2^k/64,2^l/64},$$

and analogously for the right hand sides. The extrapolation coefficients have been determined with the computer algebra program in Rüde [14] such that all error terms $h_x^{2k} h_y^{2l}$ in the multivariate expansion with $1 \leq k, l \leq 4$ are eliminated. Table 3 summarizes the results for this problem for varying penalty coefficient (and exact enforcement of the constraint). The importance of eliminating the spurious modes can be seen clearly. With the constraint, the method is equivalent to p-elements of degree 3.

Note at last that the combination technique applied to the functional leads to a method that coincides with the sparse grid finite element technique introduced in section 3 for constant coefficient operators.

Table 3
High order energy extrapolation with constraints for a smooth model problem

ϵ	0	1	10	100	1000	∞	interp.
L_2	1.872e-5	1.502e-5	1.371e-6	4.796e-7	3.946e-7	3.934e-7	2.781e-7

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