UPSCALING TECHNIQUES BASED ON SUBSPACE CORRECTIONS AND COARSE-GRID APPROXIMATIONS[1]

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ABSTRACT

In this paper we consider new upscaling and homogenization methods for the pressure equation that are less sensitive to the boundary condition assumptions than many other published methods. We start from the discretized pressure equation and rely on the construction of subproblems with significantly fewer unknowns, describing the behaviour of the solution of the fine-grid problem in suitable subspaces of the finite-element space. We utilize ideas known from multilevel methods such as operator-dependent interpolation and we clarify the part of Schur-complement approximations in the context of upscaling. We sketch several schemes for recursive approximation of the Schur-complement and report on the results of numerical experiments.

INTRODUCTION

In many cases the mathematical modelling of physical processes leads to partial differential equations with highly oscillatory coefficients. Typical examples are the mechanics of disperse media, filtration, composite materials and the modelling of flow in reservoirs made of strongly heterogeneous media. Because of storage requirements and numerical complexity, a direct numerical solution of such problems is impracticable in most cases. In petroleum reservoir models for example, the basic geophysical properties are usually generated by geostatistical reservoir description algorithms. These result in fine-scale descriptions of reservoir properties, such as permeability, on grids far too fine to

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be used as a computational grid[11]. This necessitates the construction of averaged models disconnecting the mesh size of the computational grid from the size of the heterogeneities of the underlying medium. Thus, the first step in the direction of an accurate numerical treatment of these problems is the determination of an approximate mathematical model, which captures the influence of the unresolved fine scales of the underlying media. This upscaling or homogenization procedure results in equations with so-called effective coefficients, varying on a coarse scale.

In the case of single-phase upscaling, an effective model is obtained from the determination of an effective permeability tensor, resulting in the same total flow of fluid through the homogeneous medium as obtained from the heterogenous medium[11]. In the one-dimensional case it can be shown that the effective permeability tensor is given by the harmonic average[31]. However, this is no longer true in multiple dimensions. Various tools have been developed to determine macroscopic models and effective permeabilities, such as simplistic means (arithmetic, geometric or harmonic), effective medium theory[25] and power-law averaging[16]. These methods are computationally efficient, but suffer from severe limitations, especially in the case of strong permeability variations[9,16,25]. Hierarchical methods such as renormalization[25] offer an efficient and more accurate approach, but they are also unreliable for a large class of problems especially with significant transport at an angle to the grid lines, as the homogenized permeability tensor is always a diagonal tensor. This together with the boundary conditions applied in the renormalization procedure may lead to significant errors in the computed flows. Direct methods such as the pressure-solver methods described for example in Ref. 18 give excellent results, but are computationally expensive. There local fine-scale problems have to be solved, which lead to approximations of the coarse-scale permeability tensor. Again a drawback of these approaches is the introduction of artificial boundary conditions for the local fine-scale problems, which influence the coarse-scale behaviour of the solution. Moreover in order to ensure numerically converged results, grid refinement studies, i.e., computations on sucessively finer grids, have to be performed. These further increase the cost. In many cases including periodic, almost-periodic and random inhomogeneities, an asymptotic analysis can be applied to define the homogenized permeability tensor by rigorous mathematical means. We refer the interested reader to the vast literature on the subject of homogenization. See for example Refs. 6, 23, 24 and references therein.

In pressure-solver methods the pressure equation has to be solved locally subject to some boundary conditions. In general an analytic solution to these equations is not available and numerical solution algorithms have to be applied. As the direct solution of the discretized equations (using Gauss elimination or LU-factorization) is far too expensive, iterative methods are used to compute an approximation to the solution. Well-known methods for the solution of the linear systems include the Jacobi, Gauss-Seidel and the related successive over-relaxation (SOR) methods[21]. More refined techniques are the multilevel preconditioned conjugate gradients and multigrid methods; see Refs. 10,21,22 and references therein. The effect of Jacobi or Gauss-Seidel iterations on symmetric positive-definite matrices is well understood. The error – after a sufficient number of iterations – is smooth and can thus be well approximated by means of a coarser grid. There exist methods to determine coarse-grid equations and the interpolation involved in the construction of the coarse-grid equations in an *operator-dependent* way[13,14,19,28,33]. See Ref. 26 for a general framework. This results in "physically meaningful" coarse-grid operators, which can be viewed as averaged or discrete homogenized equations. This observation lead to the development of so-called multigrid-homogenization methods for the computation of effective permeability tensors[15,20,26]. These hierarchical methods start from a discretized form of the pressure equation and construct coarse-grid equations using operator-dependent interpolation and variational coarsening[10]. An approximation to the effective permeability tensor is then extracted from the coarse-grid equations using either physically motivated arguments[15] or by expanding the coarse-grid operator in a local basis of discrete differential operators[20,26]. These methods proved to be reliable for a variety of cases, including problems with random permeability fields with very large jumps and problems with significant transport at an angle to the grid lines. However, in their present form they are not directly applicable to the general case of nonperiodic permeabilities.

To state it again, the drawbacks of the direct methods are the cost of the method and the introduction of artificial boundary conditions. However, operator-dependent interpolation and variational coarsening as applied in the multigrid homogenization methods provide a way of constructing discrete coarse-grid problems with significantly fewer unknowns whose solution leads to approximations of the solution of the fine-scale equations and thus to approximations of the effective permeability tensor.

The aim of this paper is to define an upscaling method that has the relatively low cost of hierarchical methods and that is less sensitive to the boundary conditions applied than the standard pressure-solver methods. Hence we present the idea of combining the method of variational coarsening with the direct pressure-solver methods. Instead of trying to solve the pressure equations directly with iterative methods, we recursively construct coarse-grid problems with significantly fewer unknowns. This can equivalently be considered as approximation of the solution of the fine-grid equations in certain subspaces of the finite-element space used for discretization. Because of the lower dimensionality these subproblems can be solved efficiently. We show that the coarse-grid operator should approximate the Schur-complement of the fine-grid equations. We outline methods for the approximation of the Schur-complement and give numerical examples. The resulting upscaling methods are closely related to operator-dependent variational coarsening as it is applied in multilevel methods and to the multigrid-homogenization methods. The method presented here is, however, also applicable in the case of nonperiodic permeabilities, which is in contrast to the multigrid-homogenization methods from Refs. 15, 20, 26.

The outline of the remainder of the paper is as follows. In the first section, we introduce the model problem, derive its discretization with finite elements and review homogenization and direct pressure-solver methods for the model problem. Then we show that variational coarsening can be interpreted as discrete upscaling, and we give some remarks on the construction of the coarse-grid equations and give the upscaling method. Then we discuss recursive approximation schemes for the Schur-complement. Finally, results of numerical experiments illustrating the theory of the earlier sections are given.

UPSCALING TECHNIQUES BASED ON SUBSPACE CORRECTIONS AND COARSE-GRID APPROXIMATIONS

In this section we introduce the model problem, discuss its discretization by finite elements and give some remarks on homogenization and pressure-solver methods as far as is necessary for the understanding of the following. For a thorough introduction into the subject we refer the reader to the literature given in the Introduction.

The Model Problem and Its Discretization

Consider the model problem

$$\text{Find } u \in X \quad \text{where} \quad \begin{array}{cl} -\nabla(K\nabla u) = f & \text{in } \Omega = [0,1]^d, \quad K(x) \in \mathbb{R}^{d\times d}, \\ B.C. & \text{on } \partial\Omega \end{array} \tag{1}$$

with $f \in L^2(\Omega)$ and

$$\begin{array}{rcl} K : \Omega & \longrightarrow & \mathbb{R}^{d\times d} \\ x & \longmapsto & K(x) = (a_{ij})_{ij} \end{array}$$

bounded, symmetric and positive definite, X a suitable solution subspace. Suppose any sensible boundary conditions $(B.C.)$ so that (1) is well posed. It is a simplified mathematical model for single-phase saturated porous-media flow that results from Darcy's law[1]. Then K is the permeability or diffusion tensor, which is in general not continuous and highly varying on several length scales.

Discretizing (1) with finite elements using nodal basis functions on a regular grid leads to a a linear system of equations

$$A_k x_k = b_k, \tag{2}$$

where A_k is a sparse positive-definite matrix and contains 3^d entries per row and column (away from the boundary)[21]. Alternatively Eq. (2) can be reformulated as minimization of

$$E_h(x) = \frac{1}{2} x^t A_k x - x^t b_k \tag{3}$$

in the space $\mathbb{R}^{n_k}$.

The Problem of Effective Permeability

In general a fine-scale computation of (1) is intractable and hence an *effective* model suitable for efficient computation has to be obtained. The objective is to find an effective permeability tensor $\tilde{K}$ (slowly varying) so that (1) with

K substituted by $\tilde{K}$ is tractable to efficient computation and that the solution of this new problem is in some sense "near" to the solution of the original problem, i.e., the gross features of the flow are preserved.

In homogenization or pressure-solver methods for the definition of coarse-scale models, (1) has to be solved locally, subject to some specified boundary conditions. The solution of these local problems then leads to approximations of the effective permeability coefficient [6, 24]. Let us review the basic facts. Consider a pressure-solver method for (1). One solves the pressure equations

$$\begin{array}{rcl} -\nabla(K\nabla u) & = & 0 \\ B.C. & & \end{array} \tag{4}$$

locally on a domain Y, with d different boundary conditions, for example constant Dirichlet boundary conditions on two opposite faces and no flow[11] or periodic[18] boundary conditions on the other sides. Assume for example $u = 1$ and $u = 0$ on two opposite faces. The coarse gridblock permeability is then determined such that the total flow rate is preserved. Assuming approximate solutions $w_1, \ldots, w_d$ of Eq. (4) on Y, we let $W = \begin{pmatrix} \nabla w_1, & \cdots, & \nabla w_d \end{pmatrix}$. An approximation to the effective permeability tensor is then obtained from

$$K^{ps} = \int_Y W(y)^t K(y) W(y) dy. \tag{5}$$

Let A_k denote the matrix of the linear system associated with the finite-element approximation of Eq. (4) on Y without boundary conditions and let $(x^1, \ldots, x^d)$ denote the discrete solutions of the local problems, Eq. (4), on a uniform grid Ω_k. Then the numerical approximation of Eq. (5) can be computed without numerical differentiation from

$$K^{ps} = (x^1, \ldots, x^d)^t A_k (x^1, \ldots, x^d). \tag{6}$$

Recursive application of this procedure leads to an approximation of the effective permeability tensor on Ω. Similar procedures are used in the homogenization (asymptotic analysis) methods. For more details we refer the reader to the literature given in the introduction.

THE UPSCALING METHOD

The idea of the upscaling method presented in this section is to approximate the discretized version of Eq. (4) by a problem with significantly fewer unknowns. The solution of this small problem then leads via Eq. (6) to an approximation of the effective permeability tensor on Y. Moreover because of the smallness of the problem approximating Eq. (4) Y can be chosen as the domain Ω and no recursion has to be applied to compute an approximate effective permeability coefficient. This significantly reduces the influence of boundary conditions, which would have to be applied in every recursion step. Hence we will describe in the following how equations with fewer unknowns can be constructed. Then we give the upscaling method.

Suppose a pressure equation of the type of (1) – using some boundary conditions – has been discretized on a grid Ω_k with meshwidth h_k. The objective is to find a numerical approximation to the solution of Eq. (4) on a coarse grid $\Omega_l \subset \Omega_k$ by solving a subproblem

$$A_l x_l = b_l \text{ where } A_l \in \mathbb{R}^{n_l \times n_l}, b_l \in \mathbb{R}^{n_l}, n_l << n_k. \tag{7}$$

<u>Variational Coarsening is Discrete Homogenization</u>

Assume that for the discretization of the problem, a sequence of nested grids

$$\Omega_0 \subset \Omega_1 \subset \ldots \subset \Omega_{k-1} \subset \Omega_k$$

is given on Ω. We restrict ourselves to $\Omega = [0,1]^d$ and to standard coarsening, i.e., the meshwidth of Ω_l is $h_l = 2^{-l}$ and the coarsest grid Ω_0 consists of the corner points of $[0,1]^d$. Associated to it is a sequence of finite-element spaces

$$V_0 \subset V_1 \subset \ldots \subset V_{k-1} \subset V_k$$

with dimensions $n_l := dim(V_l), l = 0, 1, \cdots, k$. Here 0 denotes the coarsest and k the finest level of discretization. Consider furthermore the sequence

$$N_0 \subset N_1 \subset \ldots \subset N_{k-1} \subset N_k$$

of sets of grid points N_l in $\Omega_l, l = 0, 1, \cdots, k$.

Then the optimal way to define the subproblem, Eq. (7), is by variational coarsening[10]: Let the inter-grid transfer operator $P_l^k : \mathbb{R}^{n_l} \to \mathbb{R}^{n_k}$ (prolongation) be linear, injectiv, with $span(P_l^k) = V_l$ and consider the restatement of the linear system as minimization of the energy, Eq. (3). An approximation to the solution of Eq. (2) is then derived from a minimization of the discrete energy functional

$$min_{y \in \mathbf{R}^{n_l}} \{ \frac{1}{2} y^t (P_l^k)^t A_k P_l^k y - y^t (P_l^k)^t b_k \}.$$

Hence we have the equivalent linear system $A_l y = b_l$ where

$$A_l = (P_l^k)^t A_k P_l^k, \quad \text{and} \quad b_l = (P_l^k)^t b_k.$$

Because of $V_l \subset V_k \; \forall \; l$ with $l < k$, P_l^k can be defined recursively via

$$P_l^k = P_{k-1}^k P_{k-2}^{k-1} \cdot \ldots \cdot P_l^{l+1}.$$

Hence the operators A_l from Eqs. (7) are defined recursively via

$$A_l = (P_l^{l+1})^t A_{l+1} P_l^{l+1}. \tag{8}$$

Note that the coarse-grid operators of the subproblems are uniquely determined by the fine-grid stiffness matrix A_k and the interpolation matrices $P_{l-1}^l, l = 1, \ldots, k$. Then the resulting coarse-grid operators can be viewed as averaged or homogenized operators.

The Upscaling Method

As an example we describe our method using constant Dirichlet boundary conditions on two opposite faces and periodic boundary conditions on the others. Note, however, that the method is not limited to this special case but is applicable with other boundary conditions as well.

Let A_k denote the matrix of the linear system associated with the finite-element approximation of (1) without boundary conditions. Then we have the following upscaling method:

Upscaling from Ω_k to Ω_l, $l < k$:

1. Compute $A_l = P_{l+1}^l A_{l+1} P_l^{l+1}$ recursively from A_k
2. Compute $A_l^1, \ldots, A_l^d$ from A_l by incorporating the boundary conditions (periodic) and set up the d right-hand sides $b_l^1, \ldots, b_l^d$ including the Dirichlet boundary conditions

3. Solve $A_l^i x_l^i = b_l^i$, $i = 1, \ldots, d$

4. Compute an approximation to the effective permeability tensor from

$$K^{ps} = (x_l^1, \ldots, x_l^d)^t A_l (x_l^1, \ldots, x_l^d) \tag{9}$$

Note that the resulting approximations to the effective permeability tensor are in general not diagonal matrices, but are dense. The accuracy of the approximations to the effective permeability tensor arising from the solution of the coarse-grid problems depends on the choice of the prolongation and hence on the choice of the subspaces V_l only. On the coarsest level Ω_0 consisting only of the corner points of $[0,1]^d$, the solutions of the coarse-grid problems from 3. are completely determined from the Dirichlet boundary conditions. Thus the approximation, Eq. (9), can be computed directly without any additional computations (i.e., without 2. and 3. in the above scheme).

The remaining question is how to choose the finite-element subspaces V_l or equivalently the operators P_l^k. The next two subsections deal with this problem.

The Coarse-Grid Equations and the Schur-Complement

It is easy to see (see next section) that in the case of (multi-)linear interpolation for the prolongation, the resulting approximation of the effective permeability coefficient is just the arithmetic mean. As is well known, this is in general not a good approximation. Therefore we do not use (multi-)linear interpolation, but rely on the following procedure.

We define the space V_l in such a way that the solution of the coarse-grid problem is the solution of the fine-grid problem in the grid points of Ω_l. Therefore we let

$$V_l := \operatorname{span}\{g_1, \cdots, g_{n_l}\}$$

where $g_1, \cdots, g_{n_k}$ denote the discrete *Greens-functions* of the fine-grid problem, that is $A_k g_i = e_i, i = 1, \cdots, n_k$, e_i the i-th unit vector and $g_1, \cdots, g_{n_l}$ denote the Greens-functions of the fine-grid problem associated with the grid points of Ω_l (after eventual renumbering of the grid points). In matrix notation the g_i's are given by the columns of A_k^{-1}.

Partitioning of A_k in coarse grid (Ω_l) and fine-without-coarse grid ($\Omega_k \setminus \Omega_l$) parts gives

$$\begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} b^1 \\ b^2 \end{pmatrix} \tag{10}$$

where u^1 denotes the unknowns in $\Omega_k \setminus \Omega_l$ and u^2 the unknowns in the coarser grid Ω_l. Equivalently A_k^{-1} is partitioned

$$A_k^{-1} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}. \tag{11}$$

Because of $A_k A_k^{-1} = I$ we then have

$$\begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} g_{12}g_{22}^{-1} \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ g_{22}^{-1} \end{pmatrix}.$$

Note that the columns of $\begin{pmatrix} g_{12} \\ g_{22} \end{pmatrix}$ are the Greens-functions of the fine-grid equations in coarse-grid points and hence the columns of $\begin{pmatrix} g_{12}g_{22}^{-1} \\ I \end{pmatrix}$ are linear combinations of these Greens-functions which span V_l. With

$$P_l^k := \begin{pmatrix} g_{12}g_{22}^{-1} \\ I \end{pmatrix} \tag{12}$$

we have

$$A_l = (P_l^k)^t A_k P_l^k = \begin{pmatrix} g_{12}g_{22}^{-1} \\ I \end{pmatrix}^t \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} g_{12}g_{22}^{-1} \\ I \end{pmatrix} = g_{22}^{-1}, \tag{13}$$

that is, the coarse-grid operator A_l is defined as the inverse of the lower right block of A_k^{-1}. We note that with this choice we have indeed

$$A_l x_l = b_l \Longrightarrow x_l = g_{22}\begin{pmatrix} g_{22}^{-1}g_{21} & I \end{pmatrix} b_k = \begin{pmatrix} g_{21} & g_{22} \end{pmatrix} b_k = \begin{pmatrix} 0 & I \end{pmatrix} A_k^{-1} b_k = x^2. \tag{14}$$

It is easy to see that

$$A_l = g_{22}^{-1} = A^{22} - A^{21}(A^{11})^{-1}A^{12}, \tag{15}$$

A_l is the well-known Schur-complement as it arises in the block Gaussian elimination process

$$\begin{pmatrix} I & -(A^{11})^{-1}A^{12} \\ 0 & I \end{pmatrix}^T \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix} \begin{pmatrix} I & -(A^{11})^{-1}A^{12} \\ 0 & I \end{pmatrix} \tag{16}$$

$$= \begin{pmatrix} A^{11} & 0 \\ 0 & A^{22} - A^{21}(A^{11})^{-1}A^{12} \end{pmatrix}. \tag{17}$$

Moreover we have

$$P_l^k = \begin{pmatrix} -(A^{11})^{-1}A^{12} \\ I \end{pmatrix}. \tag{18}$$

That is, the upscaling procedure is just a Gaussian elimination process. Note that many other upscaling methods including the renormalization approach of[25] can also be considered as approximate Gaussian elimination processes.

RECURSIVE SUBSPACE APPROXIMATION SCHEMES

In this section we construct approximations to the coarse-grid operators defined in the preceding section. Obviously it is too expensive in general to compute the coarse-grid operator A_l from Eq. (15) exactly, as this would involve the computation of $(A^{11})^{-1}$ and moreover in general the sparsity structure of A_k is lost. In the following we restrict ourselves to 1- and 2D. There are a variety of methods to construct these approximations and to perform an approximate Gaussian elimination; see for example Refs. 12–14, 19, 28, 29 and Refs. 3, 4, 32, 33. We sketch the idea in great generality. Most of this section is taken from Ref. 26. See there for more details.

Standard Coarsening With (Bi-)Linear Interpolations

For completeness we consider first the case of (bi-)linear interpolation, which is equivalent to letting V_l be spanned by the nodal basis functions on grid Ω_l. That is, P_{l-1}^l is not chosen operator-dependent as defined in the preceding section, but is fixed. This leads to an arithmetic averaging of the permeability.

Assume that the sequence of nested grids is uniform and equidistant with mesh sizes $h_l = 2^{-l}, l = 1, \cdots, k$. Let $e_i^l, e_{i,j}^l$ denote the vectors in $\mathbb{R}^{n_l}$ that are one in the value corresponding to the i-th (1D) or (i,j)-th (2D) grid point, respectively, and zero elsewhere. Let P_{l-1}^l be defined by

$$P_{l-1}^l e_{IC}^{l-1} = \frac{1}{2} e_{IF-1}^l + e_{IF}^l + \frac{1}{2} e_{IF+1}^l$$

in 1D (see Figure 1) and

$$P_{l-1}^l e_{IC,JC}^{l-1} = \frac{1}{4} e_{IF-1,JF-1}^l + \frac{1}{2} e_{IF-1,JF}^l + \frac{1}{4} e_{IF-1,JF+1}^l$$

$$+\frac{1}{2}e^l_{IF,JF-1} + e^l_{IF,JF} + \frac{1}{2}e^l_{IF,JF+1}$$
$$+\frac{1}{4}e^l_{IF+1,JF-1} + \frac{1}{2}e^l_{IF+1,JF} + \frac{1}{4}e^l_{IF+1,JF+1}$$

in 2D (see Figure 2), i.e., the inter-grid transfer operators are defined as the linear and bilinear interpolation in 1D and 2D, respectively.

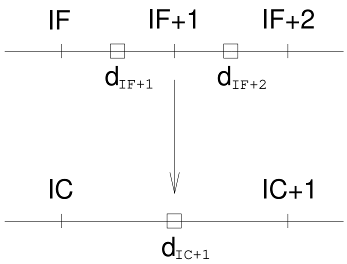

Fig. 1: 1D Case: Fine and Coarse Grid

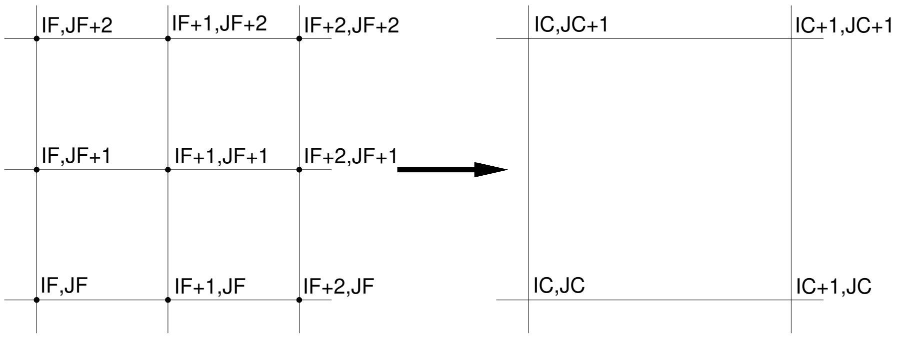

Fig. 2: 2D Case: Fine and Coarse Grid

This can be conveniently written in stencil notation (See Refs. 21, 33)

$$(P^l_{l-1})^* = \frac{1}{2}\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \qquad \text{(1D case)}$$

$$(P^l_{l-1})^* = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \qquad \text{(2D case).}$$

Then, the underlying basis for each subspace $V_l, l = 1, \cdots, k$ is the nodal basis defined by the grid $\Omega_l, l = 0, 1, \ldots, k$.

It is easy to see that, with this choice of inter-grid transfer operators, an arithmetic averaging of the permeability tensor is performed. Consider for example the 1D pressure equation $-(d(x)u(x)_x)_x = f(x)$, where $d(x)$ is assumed to be piecewise constant on the finest grid Ω_k, having jumps only at the grid points. We obtain by discretizing with finite elements on Ω_k for a fine-grid point $IF+1$ the equation

$$\frac{1}{h_k}(-d^k_{IF+1}x_{IF} + (d^k_{IF+1} + d^k_{IF+2})x_{IF+1} - d^k_{IF+2}x_{IF+2}) = (b_k)_{IF+1}.$$

See Figure 1.

From the variational coarsening, Eq. (8), we obtain as a coarse-grid equation in $IC \in \Omega_{k-1}$

$$\frac{1}{h_{k-1}}\left(-\frac{d^k_{IF-1} + d^k_{IF}}{2}x_{IC-1} + \frac{(d^k_{IF-1} + d^k_{IF} + d^k_{IF+1} + d^k_{IF+2})}{2}x_{IC} \right.$$
$$\left. - \frac{d^k_{IF+1} + d^k_{IF+2}}{2}x_{IC+1}\right) = \frac{1}{4}(b_k)_{IF-1} + \frac{1}{2}(b_k)_{IF} + \frac{1}{4}(b_k)_{IF+1}.$$

That is, an arithmetic mean ($d^{k-1}_{IC} = \frac{d^k_{IF-1}+d^k_{IF}}{2}$) is computed implicitly.

Standard Coarsening With Matrix-Dependent Interpolations

Assume again standard coarsening of the grids. We will now define the inter-grid transfer operators as in Eq. (13). As already mentioned, the computation of $(A^{11})^{-1}$ is in general too expensive. Moreover $(A^{11})^{-1}$ is in general not a local operator, and therefore the sparsity structure of A_{l-1} is lost.

However, in the 1D case things are different [22]. The partitioning of the associated matrix according to Eq. (10) now results in just a diagonal matrix A_{11}. Then $(A^{11})^{-1}$ is diagonal and consequently the prolongation P^l_{l-1} and the coarse-grid operator A_{l-1} are local operators.

From Eq. (18) we derive the interpolation

$$P^k_{k-1}e^k_{IC} = \frac{d^k_{IF}}{d^k_{IF-1} + d^k_{IF}}e^{k-1}_{IF-1} + e^{k-1}_{IF} + \frac{d^k_{IF+1}}{d^k_{IF+1} + d^k_{IF+2}}e^{k-1}_{IF+1}. \qquad (19)$$

From Eq. (15) we obtain the coarse-grid equation

$$\frac{1}{h_{k-1}}\left(-\frac{2d^k_{IF-1}d^k_{IF}}{d^k_{IF-1} + d^k_{IF}}x_{IC-1} + \left(\frac{2d^k_{IF-1}d^k_{IF}}{d^k_{IF-1} + d^k_{IF}} + \frac{2d^k_{IF+1}d^k_{IF+2}}{d^k_{IF+1} + d^k_{IF+2}}\right)x_{IC} \right.$$
$$\left. - \frac{2d^k_{IF+1}d^k_{IF+2}}{d^k_{IF+1} + d^k_{IF+2}}x_{IC+1}\right) =$$

$$= \quad \frac{1}{2}\frac{d^k_{IF}}{d^k_{IF-1}+d^k_{IF}}(b_k)_{IF-1} + \frac{1}{2}(b_k)_{IF} + \frac{1}{2}\frac{d^k_{IF+1}}{d^k_{IF+1}+d^k_{IF+2}}(b_k)_{IF+1}.$$

That is, the harmonic mean of the two corresponding fine-grid values is computed implicitly in the coarse-grid operator. Applying this approach recursively for the sequence of successively coarser grids, we finally obtain on Ω_1 the global harmonic mean. This is the exact result of homogenization[31].

In the two-dimensional case A^{11} is tridiagonal and the exact operator-dependent interpolation cannot be used. Therefore approximations to the operator-dependent interpolation are used, which result in local operators. Hence a local approximation M of $A_{11}^{-1}A_{12}$ has to be found such that the error $||A_{11}^{-1}A_{12} - M||$ in some norm is as small as possible. The following approach can be used[12, 19, 26, 28]. Substitute the set of equations associated to the grid points $\Omega_k \setminus \Omega_{k-1}$ by an altered set of equations, i.e., substitute $\left(\begin{array}{cc} A^{11} & A^{12} \end{array} \right)$ by some $\left(\begin{array}{cc} \tilde{A}^{11} & \tilde{A}^{12} \end{array} \right)$, where $\tilde{A}^{11}$ is 'easy' to invert, $(\tilde{A}^{11})^{-1}\tilde{A}^{12}$ is a local operator and is an approximation of $(A^{11})^{-1}A^{12}$. The operator $\tilde{P}^k_{k-1}$ is defined by

$$\tilde{P}^k_{k-1} = \left(\begin{array}{c} -(\tilde{A}^{11})^{-1}\tilde{A}^{12} \\ I \end{array} \right) \tag{20}$$

and the resulting coarse-grid operator A_{k-1} is given by

$$A_{k-1} = (\tilde{P}^k_{k-1})^t A_k \tilde{P}^k_{k-1}.$$

We consider one example how to construct an approximate operator-dependent interpolation. If $l > 0$, then every grid Ω_l can be divided into four disjunct grids:

$$\begin{array}{lcl} \Omega_l^{00} & := & \{(x,y) \in \Omega_l : x = ih_l, y = jh_l, i,j \in \mathrm{Z}, i \text{ even}, j \text{ even}\} = \Omega_{l-1} \\ \Omega_l^{10} & := & \{(x,y) \in \Omega_l : x = ih_l, y = jh_l, i,j \in \mathrm{Z}, i \text{ odd}, j \text{ even}\} \\ \Omega_l^{01} & := & \{(x,y) \in \Omega_l : x = ih_l, y = jh_l, i,j \in \mathrm{Z}, i \text{ even}, j \text{ odd}\} \\ \Omega_l^{11} & := & \{(x,y) \in \Omega_l : x = ih_l, y = jh_l, i,j \in \mathrm{Z}, i \text{ odd}, j \text{ odd}\}. \end{array} \tag{21}$$

Consider the following substitutions, which involve 'directional' lumping of fine-grid stencils[13]. Let

$$\left[\begin{array}{ccc} L_{-1,1} & L_{0,1} & L_{1,1} \\ L_{-1,0} & L_{0,0} & L_{1,0} \\ L_{-1,-1} & L_{0,-1} & L_{1,-1} \end{array} \right]$$

denote a stencil of the fine-grid operator A_k. The stencils in points of Ω^{10} and Ω^{01} are changed in the following way:

$$\begin{bmatrix} L_{-1,1} & L_{0,1} & L_{1,1} \\ L_{-1,0} & L_{0,0} & L_{1,0} \\ L_{-1,-1} & L_{0,-1} & L_{1,-1} \end{bmatrix} \longrightarrow \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline L_{-1,0} & L_{0,0} & L_{1,0} \\ +L_{-1,-1} & +L_{0,-1} & +L_{1,-1} \\ +L_{-1,1} & +L_{0,1} & +L_{1,1} \\ \hline 0 & 0 & 0 \end{array}\right] \text{ in } \Omega^{10} \tag{22}$$

$$\begin{bmatrix} L_{-1,1} & L_{0,1} & L_{1,1} \\ L_{-1,0} & L_{0,0} & L_{1,0} \\ L_{-1,-1} & L_{0,-1} & L_{1,-1} \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & L_{0,1}+L_{-1,1}+L_{1,1} & 0 \\ 0 & L_{0,0}+L_{-1,0}+L_{1,0} & 0 \\ 0 & L_{0,-1}+L_{-1,-1}+L_{1,-1} & 0 \end{bmatrix} \text{ in } \Omega^{01}. \tag{23}$$

With this choice, the resulting coarse-grid operator A_{l-1} and the associated P^l_{l-1} by means of the variational coarsening, Eq. (13), do have again 9 nonzero entries per row and the operator-dependent coarsening procedure can be repeated recursively on coarser grids. This method is also utilized in the multigrid-homogenization methods introduced in Refs. 15, 26.

We did not care about the choice of the coarse grids, but used uniform, equidistant and nested grids. The subspaces chosen from V_k were then spanned by the discrete Greens-functions associated to the coarse-grid points (1D) or by approximations to these (2D). However, it is clear that the choice of the coarse grids and thus the spaces $V_l, l = 1, \cdots, k-1$ is of importance for the accuracy of the approximation of the fine-grid solution by the coarse-grid solution. This is the case, for example, for strongly anisotropic (effective) operators or equivalently for problems with a large aspect ratio of the grids. These result in Schur-complements, which are dense matrices that cannot be approximated directly by sparse matrices. Therefore different coarsening techniques, like semi-coarsening (coarsening with respect to only one coordinate direction), have to be applied to regain good approximations of the Schur-complement. Another way for a further improvement of the schemes can possibly be derived by so-called algebraic coarsening methods as described for example in Ref. 30. There also the selection of the coarse grids is determined from the fine-grid operator. Another related and promising way for the computation of discrete homogenized operators is presented in Refs. 7, 8, 17. There the transformation in a wavelet basis, compression of the resulting operators and computation of the Schur-complement is used. However, there are no results for comparisons yet.

NUMERICAL EXPERIMENTS

We now consider two-dimensional problems with different types of permeability fields and report on the results of the upscaling method described in the preceding sections. Various simulations with different coefficients have been run. We present only three such simulations to illustrate the results. We will compare the results to the classical averaging (arithmetic, harmonic and geometric) used in petroleum engineering and to the results of asymptotic analysis and renormalization. The results of asymptotic analysis were computed using conforming finite elements. We used the scheme of Dendy[13] to compute approximations to the exact operator-dependent prolongations and coarse-grid operators. See the above sections. In the following the results of our method will be denoted by MG-homogenization.

As shown above the method is exact for one-dimensional problems. Moreover, for a layered medium in 2D with the layering aligned with the coordinate axes, again the exact result (the harmonic mean in the one and the arithmetic mean in the other coordinate direction) is obtained with our method. This is due to the one-dimensional nature of the problem. See Ref. 20 and Ref. 15 for a longer discussion. See also there for a pathological problem (a permeability field with a checkerboard structure), where MG-homogenization using Dendy's scheme does not work better than the arithmetic mean.

Test Problem 1: Medium With Inclusions

This test problem involves the computation of the effective coefficients of a homogeneous and isotropic medium with heterogeneities.

The data are

$$K(x) = \begin{cases} \alpha \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \forall x \in Y_I \\ 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \forall x \in Y_{II}. \end{cases}$$

This is illustrated in Figure 3. The symmetry of the problem and the diagonal

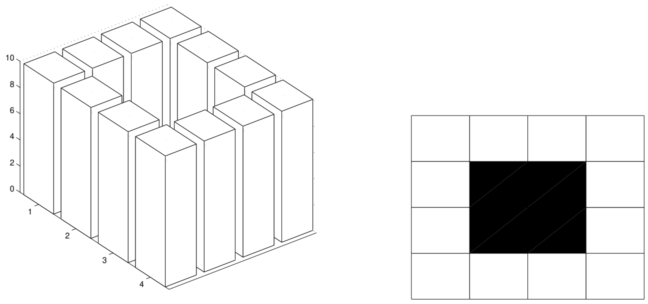

Figure 3: Permeability Field for Example 1, $\alpha = 10$

TABLE 1

Problem 1: Comparison of the Approximations of the Effective Permeabilities

α	Asymptotic Anal.	MG-homogen.	Renormalization	Arith. mean
0.01	$1.71e-2$	$1.80e-2$	$1.59e-2$	$2.58e-1$
0.1	$1.55e-1$	$1.60e-1$	$1.47e-1$	$3.25e-1$
1	1	1	1	1
10	6.52	6.70	6.17	7.75
100	59.20	61.74	54.25	75.25
1000	579.32	611.74	534.25	750.25

structure of K imply that the equivalent material is still isotropic. A grid of 3×3 inner grid points has been used for the discretization of the problem, and the approximation of the effective permeability coefficient has been computed on grid Ω_0. The results are summarized in Table 1. The results of MG-homogenization and of renormalization are relatively close to the ones obtained from asymptotic analysis.

Test Problem 2: Random Permeability Field

The data are:

$$K(x) = d(x) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where the values of $d(x)$ are randomly chosen over Ω and assumed to be piecewise constant on fine-grid cells. The permeability d is distributed according

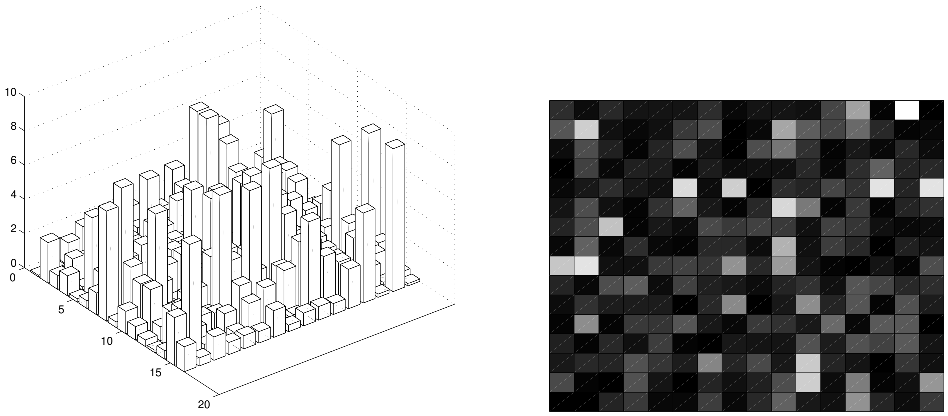

Figure 4: Logarithm of the Permeabilities for Example 2, $a = 2$

TABLE 2

Results for Random Permeabilites

a	2	5	10
max	19.05	$1.10e+4$	$5.33e+5$
min	1.00	1.00	1.01
standard dev.	2.45	745	31715
arith. mean	2.39	100	2898
harm. mean	1.64	2.63	3.18
MG-homogen.	$\begin{pmatrix} 1.93 & 0.04 \\ 0.04 & 1.96 \end{pmatrix}$	$\begin{pmatrix} 5.12 & 0.21 \\ 0.21 & 5.09 \end{pmatrix}$	$\begin{pmatrix} 9.80 & 0.14 \\ 0.14 & 10.14 \end{pmatrix}$
geo. mean	1.87	4.85	10.21

to

$$d = z^{-ln(a)}, \;\; z \in \mathcal{X}, \mathcal{X} \text{ equally distributed in } (0,1).$$

An example is given in Figure 4 for $a = 2$. For this special distribution the exact effective permeability coefficient is the geometric mean of the fine-scale permeability coefficient [27]. The finest grid had 15×15 inner grid points, and the approximation of the effective permeability coefficient has been computed on grid Ω_0. The results are summarized in Table 2. Clearly the results of MG-homogenization are closest to the geometric means. Note that MG-homogenization results in dense tensors in this case. This is due to the finite dimension of the test problems.

Test Problem 3: Correlated Random Field

The permeability map of problem 3 is a correlated random field on a 128×128 grid (see Figure 5). Starting from a random isotropic permeability field, a moving-ellipse averaging technique was used to obtain a correlated random field. We give the results of our method for no-flow boundary conditions in the computation of the effective permeability tensor.

For comparison we also give the results of arithmetic mean, harmonic mean and a pressure-solver method (also with no-flow boundary conditions) on the grid of 128×128 cells. We computed the result of the MG-homogenization method on different levels l, i.e., the 128×128 permeability coefficient is upscaled to a grid of $2^l \times 2^l$ permeability tensors where the resulting coarse-grid problem is solved exactly. From this the approximation to the effective permeability tensor is computed. The results are summarized in Table 3 and Table 4. The results of the pressure-solver method and those of the MG-homogenization method indicate the same behaviour, a strongly anisotropic flow field. Note that the result of MG-homogenization is invariant under dyadic refinements of the computational grid. That is, the result is automatically converged and no refinement studies have to be applied.

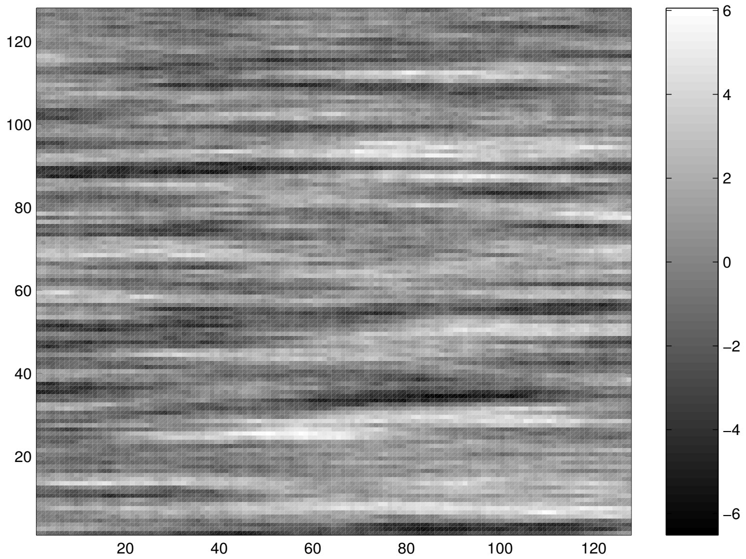

Fig. 5: Logarithm of the Permeability, Problem 3

TABLE 3

Results for Problem 3

min	max	arith. mean	harmonic mean	pressure-solver
$1.482e-3$	426.0	6.1270	0.1483	in x : 3.283 in y : 0.267

TABLE 4

Results of MG-Homogenization for Problem 3

level l (grid)	3 (8×8)	2 (4×4)	1 (2×2)
	$\begin{pmatrix} 3.677 & 0.013 \\ 0.013 & 0.312 \end{pmatrix}$	$\begin{pmatrix} 3.734 & -0.053 \\ -0.053 & 0.519 \end{pmatrix}$	$\begin{pmatrix} 3.842 & -0.233 \\ -0.233 & 0.927 \end{pmatrix}$

TABLE 5

Results of "Two-Step Upscaling" to a 2×2 Grid for Problem 3

method \ level l (grid)	4 (16×16)	3 (8×8)	2 (4×4)
MG-Homogenization	in x : 4.704 in y : 1.253	in x : 3.771 in y : 0.775	in x : 3.900 in y : 0.296
Pressure-solver	in x : 3.251 in y : 0.254	in x : 3.360 in y : 0.218	in x : 3.871 in y : 0.206

For comparison we also performed a "two-step upscaling". This is performed by upscaling the fine-scale permeability to a grid of $2^l \times 2^l$ cells and a subsequent upscaling process to the coarsest grid. The results for different levels l are summarized in Table 5. Note that for the discretization on level l that is needed for the second upscaling process, only the diagonal terms of the upscaled permeability tensor have been used.

The relatively large discrepancy between the result of "one-step" upscaling and "two-step" upscaling indicates the necessity of semi-coarsening in order to account for the strong anisotropy of the effective permeability tensor on larger scales.

CONCLUSIONS

In this paper, we have considered upscaling techniques arising from approximate solutions of the discretized pressure equation. These are related to the multigrid homogenization methods presented recently. We defined subproblems describing directly the coarse-scale behaviour of the fine-scale solution. We identified the Schur-complement as proper choice for a discrete homogenized coarse-grid operator and showed that it corresponds to the minimization of the energy functional in subspaces spanned by discrete Greens-functions and to a Gaussian elimination process applied to the fine-grid matrix. The techniques presented for the approximation of the Schur-complement are well known in the context of multilevel solvers for elliptic problems.

In the numerical tests the method gave consistently better results than renormalization or simple means. In particular it reproduced an analytic solution for a simple test case and gave good agreement with the known solution for a problem with very large jumps in a 2D random field. Moreover the method is able to capture anisotropies in the effective permeability tensor and is capable of predicting the direction of the flow with high accuracy. In standard pressure-solver methods, artificial boundary conditions have to be assumed, which may lead to significant errors in the results. The new method is less sensitive to these boundary condition assumptions. Hence the dependence of the results on these boundary conditions is greatly reduced. Unfortunately, however, their influence is not completely removed. A crucial point is that in contrast to standard methods, no grid-refinement studies have to be applied to verify that the result is numerically converged. As soon as the permeability field is completely resolved by the computational grid, computations on finer grids give the same results.

Further work needs to be done to obtain error estimates for the method and to implement semi-coarsening in order to address cases with strong anisotropy.

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