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Meshfree Multiscale Methods for Solids

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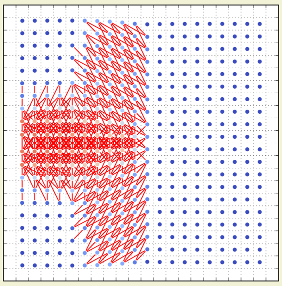
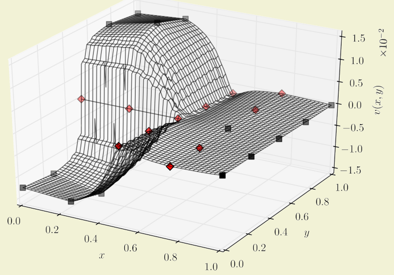
Objectives

Fast simulation of localized material failure for macroscale material samples through

- efficient (Generalized) Finite Elements for global linear elasticity problem and
- the use of solution from a local particle simulation
- as enrichment to construct discontinuous shape functions for the global problem.

Peridynamics and GFEM





Peridynamics Simulation with Particles

- Nonlocal equation of motion

$$\rho \ddot{\mathbf{u}}(\mathbf{x},t) = \int_{\Omega(\mathbf{x})} f\left(\left(\mathbf{u}(\tilde{\mathbf{x}},\cdot) - \mathbf{u}(\mathbf{x},\cdot)\right)|_{(-\infty,t]}, \tilde{\mathbf{x}} - \mathbf{x} \right) \mathrm{d}\tilde{\mathbf{x}} + b(\mathbf{x},t)$$

- No gradients, discontinuities occur naturally
- Discretization: fix $\mathbf{x_i}$, calculate $\mathbf{u_i^n} \approx \mathbf{u}(\mathbf{x_i}, t_n)$

$$\mathbf{u_i^{n+1}} = 2\mathbf{u_i^n} - \mathbf{u_i^{n-1}} + \frac{(\Delta t)^2}{\rho} \left(\sum_{j \in \mathbf{N_i}} f\left(\left(m \mapsto \mathbf{u_j^m} - \mathbf{u_i^m}\right)\Big|_{(-\infty,n]}, \mathbf{x_j} - \mathbf{x_i} \right) \mathbf{V_{i,j}} + b(\mathbf{x_i}, t_n) \right)$$

- Explicitly track bonds and bond failure between particles

$$\mathbf{A_{i,j}^{n+1}} = \begin{cases} 1 & f\left(\left(m \mapsto \mathbf{u_j^m} - \mathbf{u_i^m}\right)\Big|_{(-\infty,n]}, \mathbf{x_j} - \mathbf{x_i} \right) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Data $\mathbf{x_i}, \mathbf{u_i^{n+1}}, \mathbf{A_{i,j}^{n+1}}$ for (discontinuous) vector field approximation η^{n+1}

Moving Least Squares

- Scattered data $\mathbf{x_i}, \mathbf{u_i^{n+1}}$ approximation with

$$\eta^{n+1}(x) := q_x^{n+1}(0) \qquad q_x^{n+1} := \operatorname*{argmin}_{p \in P} \mathbf{J_x^{n+1}}(p)$$

$$\mathbf{J_x^{n+1}}(p) := \sum_i \mathbf{W_i^{n+1}}(x) \left(\mathbf{u_i^{n+1}} - p(\mathbf{x_i} - x) \right)^2$$

- $\mathbf{W_i^{n+1}}$ are normal Moving Least Squares weights W_i (some kernels) modified to take $\mathbf{A_{i,j}^{n+1}}$ into account
- Take $\mathbf{w_{i,j}}(\mathbf{x_i}) = 1, \mathbf{w_{i,j}} \leq 1$ locally supported

$$\mathbf{W_i^{n+1}}(x) := W_i(x) \prod_{\{j : \mathbf{A_{i,j}^{n+1}} = 0\}} (1 - \mathbf{w_{j,i}}(x))$$

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Acknowledgments

This work is supported and funded by the Sonderforschungsbereich 716 of the Deutsche Forschungsgemeinschaft as Subproject D.7.

Use global solution $\mathbf{u}(\cdot, t_n)$ to to choose domain $\bigcup_{i \in \mathbf{I^{n+1}}} \operatorname{supp}(\varphi_i) \ni \mathbf{x_i}$ and initial and boundary conditions (e.g. $\mathbf{u_i^n} = \mathbf{u}(\mathbf{x_i}, t_n), \ldots$) for local Peridynamics run(s)

Linear Elasticity with Generalized Finite Elements

- Local equation of motion

$$\begin{aligned} \rho \ddot{\mathbf{u}}(\mathbf{x},t) &= \mu \Delta \mathbf{u}(\mathbf{x},t) + (\lambda + \mu) \nabla \operatorname{div} \mathbf{u}(\mathbf{x},t) + b(\mathbf{x},t) \\ &= (L_t \mathbf{u}(\cdot,t))(\mathbf{x}) + b(\mathbf{x},t) \end{aligned}$$

- Partition of Unity $0 \leq \varphi_i \leq 1, i \in I \in \mathbb{N}$, suff. smooth, locally supported, $\sum_{i \in I} \varphi_i = 1$.
- Local approximation spaces P_i (polynomial(s)), $\mathbf{E_i^{n+1}} \overset{!}{\supseteq} \emptyset$ (**enrichments**) on $\operatorname{supp}(\varphi_i)$

$$E_i^{n+1} = \begin{cases} \operatorname{span}\left\{ \boldsymbol{\eta}^{\mathbf{n+1}} \right\} & i \in \mathbf{I^{n+1}} \\ \emptyset & i \notin \mathbf{I^{n+1}} \end{cases}$$

- Global shape functions for timestep t_{n+1}

$$\bigcup_{i \in I} \{\varphi_i p : p \in P_i\} \cup \bigcup_{i \in \mathbf{I^{n+1}}} \{(\varphi_i \boldsymbol{\eta}^{\mathbf{n}})\}$$

- Discretization: find coefficients $\left(c_{i,\cdot}^{n+1}\right)_{i \in I}$ and $\left(\mathbf{d_i^{n+1}}\right)_{i \in \mathbf{I^{n+1}}}$ such that

$$\begin{aligned} \mathbf{u}(\mathbf{x}, t_{n+1}) &= \sum_{i \in I} \varphi_i(\mathbf{x}) \left(\sum_{p \in P_i} c_{i,p}^{n+1} p(\mathbf{x}) \right) + \sum_{i \in \mathbf{I^{n+1}}} \mathbf{d_i^{n+1}} \left(\varphi_i \boldsymbol{\eta}^{\mathbf{n+1}} \right)(\mathbf{x}) \\ &\overset{!}{=} 2\mathbf{u}(\mathbf{x}, t_n) - \mathbf{u}(\mathbf{x}, t_{n-1}) + \frac{(\Delta t)^2}{\rho} \left((L_{t_n} \mathbf{u}(\cdot, t_n))(\mathbf{x}) + b(\mathbf{x}, t) \right) \end{aligned}$$

(Discontinuous) enrichment η^{n+1} for construction of new shape functions $\varphi_i \boldsymbol{\eta}^{n+1}$ for global elasticity problem